Gaussian

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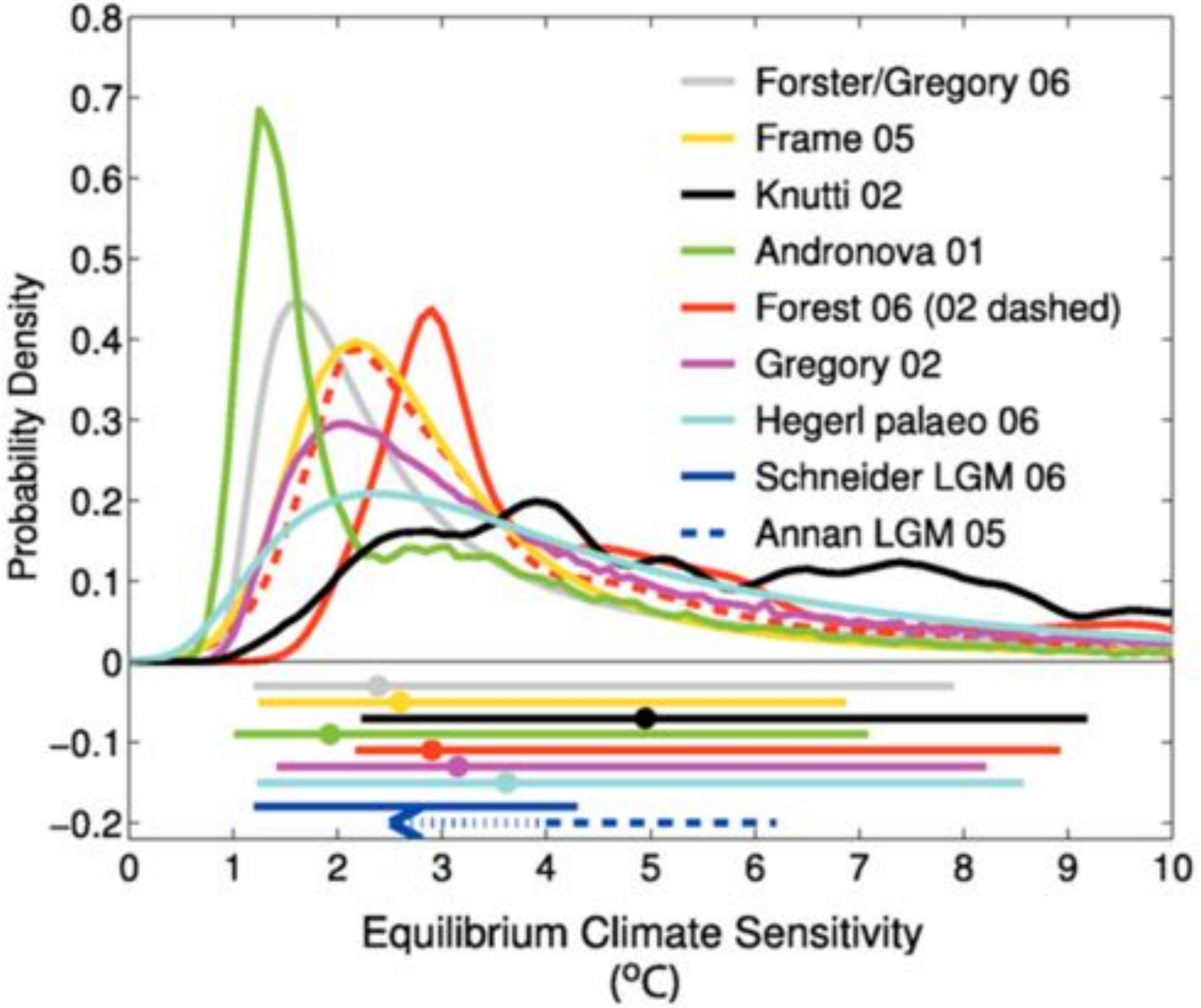
Which is Random?

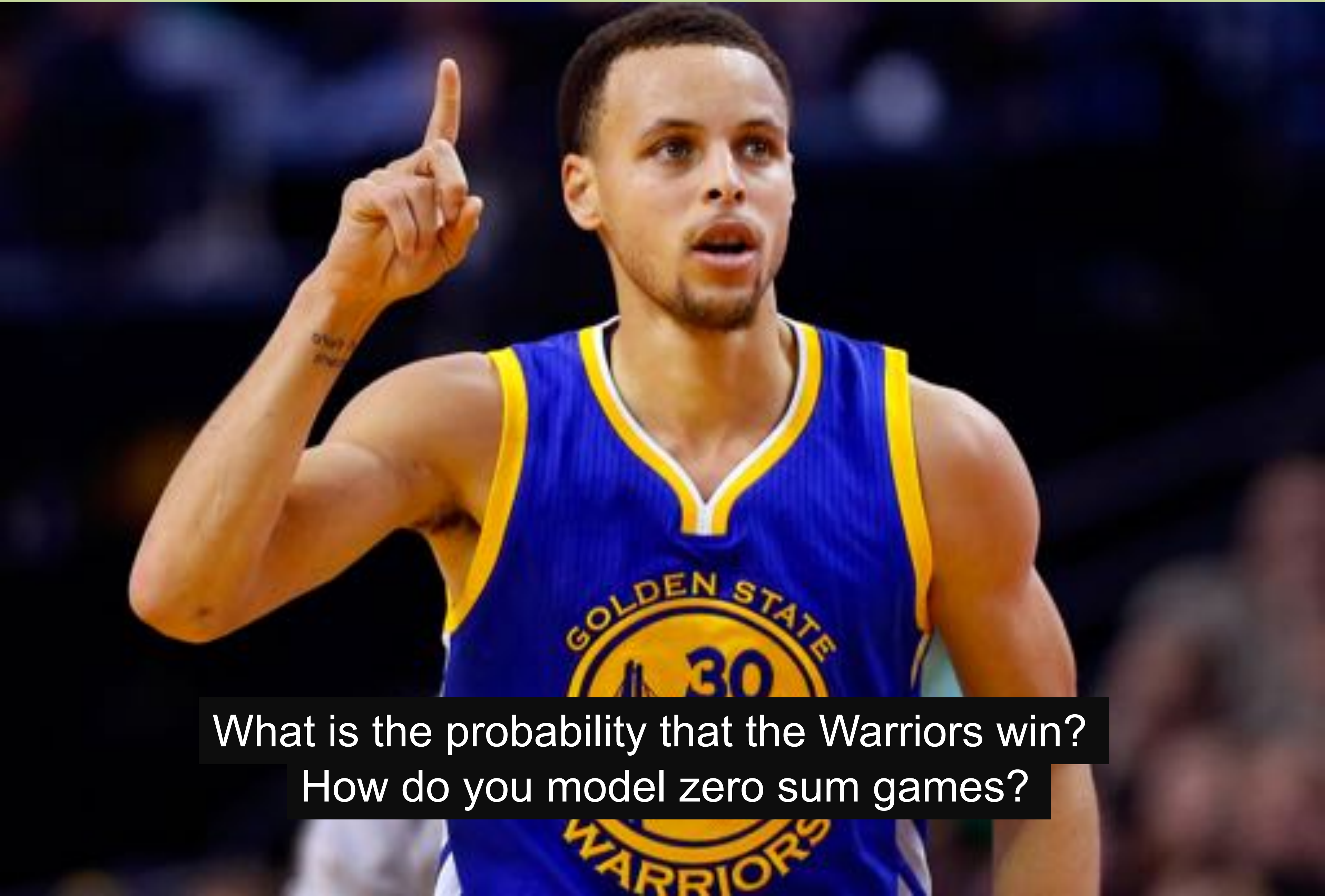
Sequence 1:
TTHHTHTTHTTTHTTTHTTTHTTHTHHTHHTHTHHTTTHHTHTHTTHTHHTTHTHHTHTTTH
HTTHHTTHHHTHHTHTTHTHTTHHTHHHTTHTHTTTHHTTHTHTHTHTHTTHTHTHHHTTHT
HTHHTHHHTHTHTTHTTHHTHTHTHTTHHTTHTHTTHHHTHTHTHTTHTTHHTTHTHHTHHH
TTHHTHTTHTHTHTHTHTHTHHHTHTHTHTTHTHHTHTHTTHTTTHHTHTTTHTHHTHHHHT
TTHHTHTHTHTHHHTTHHTHTTTHTHHTHTHTHHTHTTHTTHTHHTHTHTTT

Sequence 2:
HTHHHTHTTHHTTTTTTTTHHHTTTHHTTTTHHTTHHHTTHTHTTTTTTTHTHTTTTHHHHTH
THTTHTTTHTTHTTTTHTHHTHHHHTTTTTHHHHTHHHTTTTHTHTTHHHHTHHHHHHHHTT
HHTHHTHHHHHHTTHTHTTTHHTTTTHTHHTTHTTHTHTHTTHHHHHTTHTTTHTHTHHTT
TTHTTTTTHHTHTHHHHTTTTHTHHHHHHTHTHTHTHHHTHTTHHHTHHHHHHTHHHTHTTT
HHHTTTHHTHTTHHTHHHTHTTHTTHTTTHHTHTHTTTTHTHTHTTHTHTHT

Climate Sensitivity



Will the Warriors Win?



What is the probability that the Warriors win?
How do you model zero sum games?

Today: 3 Pineapples

Each question has a $p = 0.2$ probability of winning a pineapple

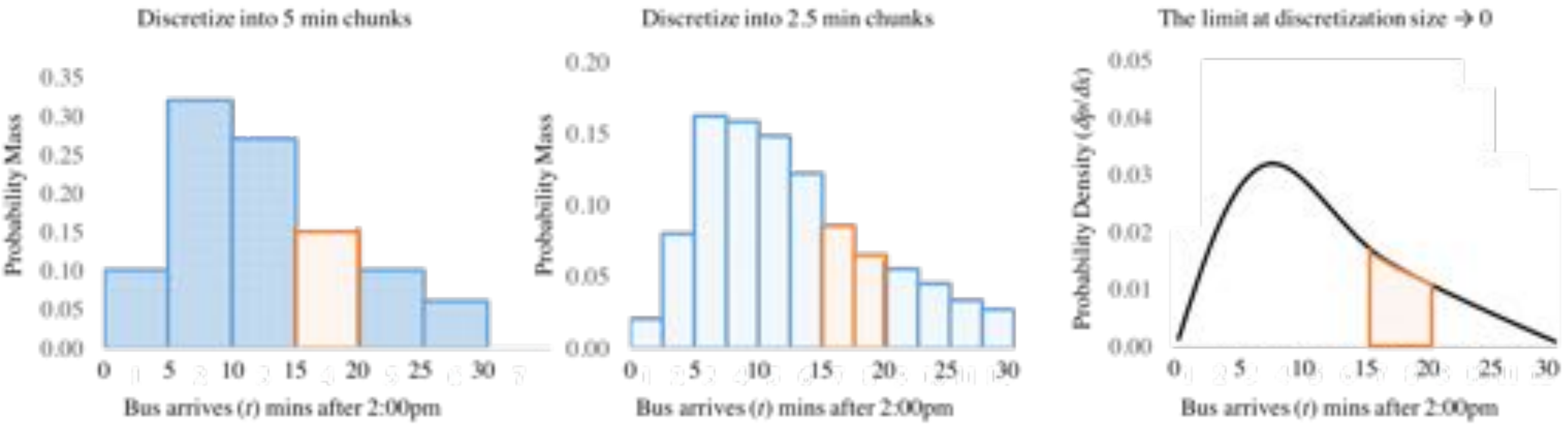


Let X be the number of trials until we are out of fruit.

$$X \sim \text{NegBin}(r = 3, p = 0.2)$$

Continuous Random Variables

Discrete to Comtinuous



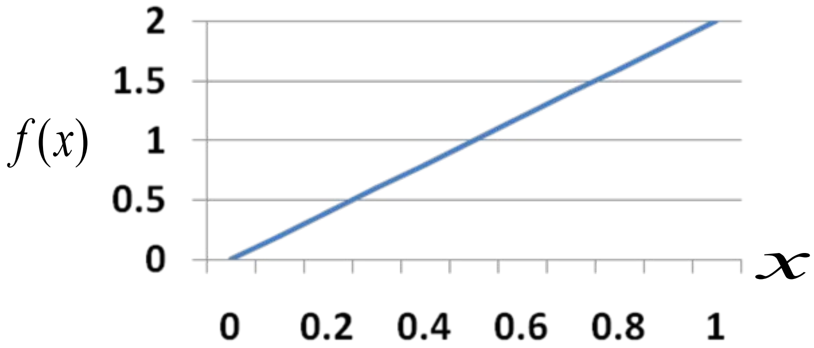
What do you get if you integrate over a probability *density* function?

A probability!

Finding Constants

- X is a continuous random variable with PDF:

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



What about $f(x) = 3x$?

$$\int_0^1 2x\ dx = x^2|_0^1 = 1$$

valid PDF

$$\int_0^1 3x\ dx = \frac{3}{2}x^2|_0^1 = \frac{3}{2}$$

Not a valid PDF

Probability Density Function

The **probability density function** (PDF) of a continuous random variable represents the relative likelihood of various values.

Units of probability *divided by units of X*.
Integrate it to get probabilities!

$$P(a < X < b) = \int_{x=a}^{b} \boxed{f_X(x)}\, dx$$

Cumulative Distribution Function

A cumulative distribution function (CDF) is a "closed form" equation for the probability that a random variable is less than a given value

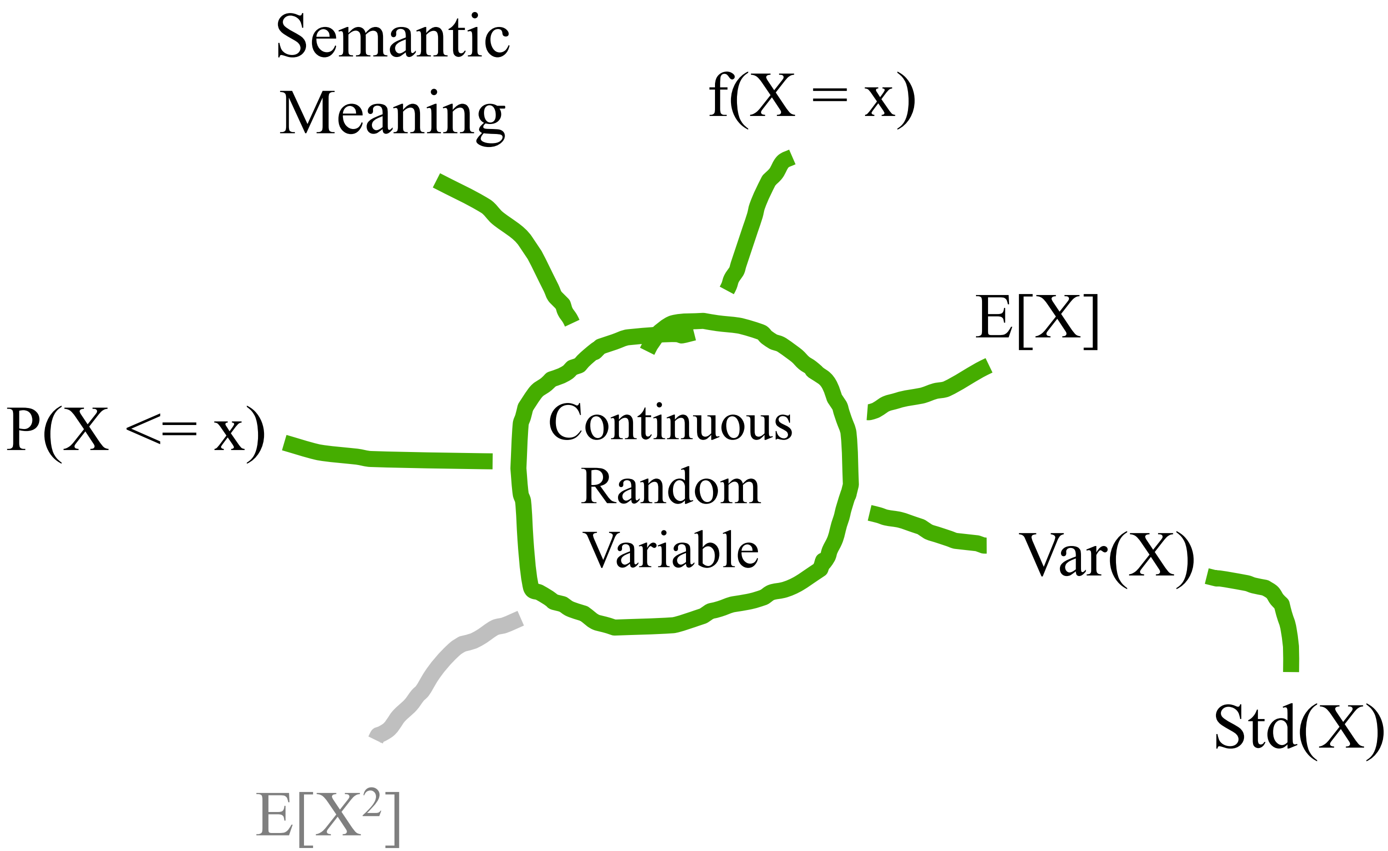
$$F(x) = P(X < x)$$

If you learn how to use a cumulative distribution function, you can avoid integrals!

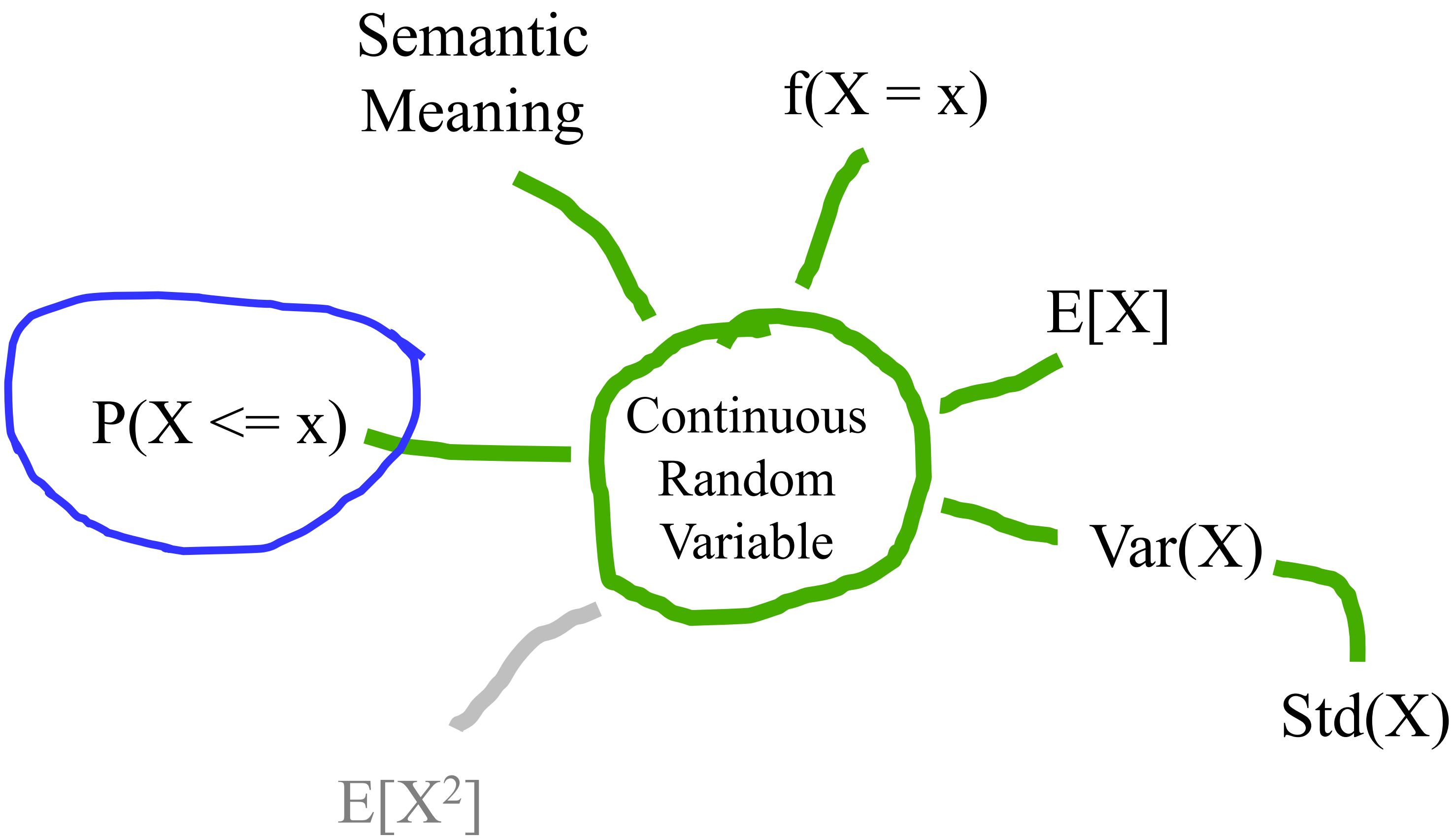
$$F_X(x)$$

This is also shorthand notation for the CDF

Fundamental Properties



Fundamental Properties



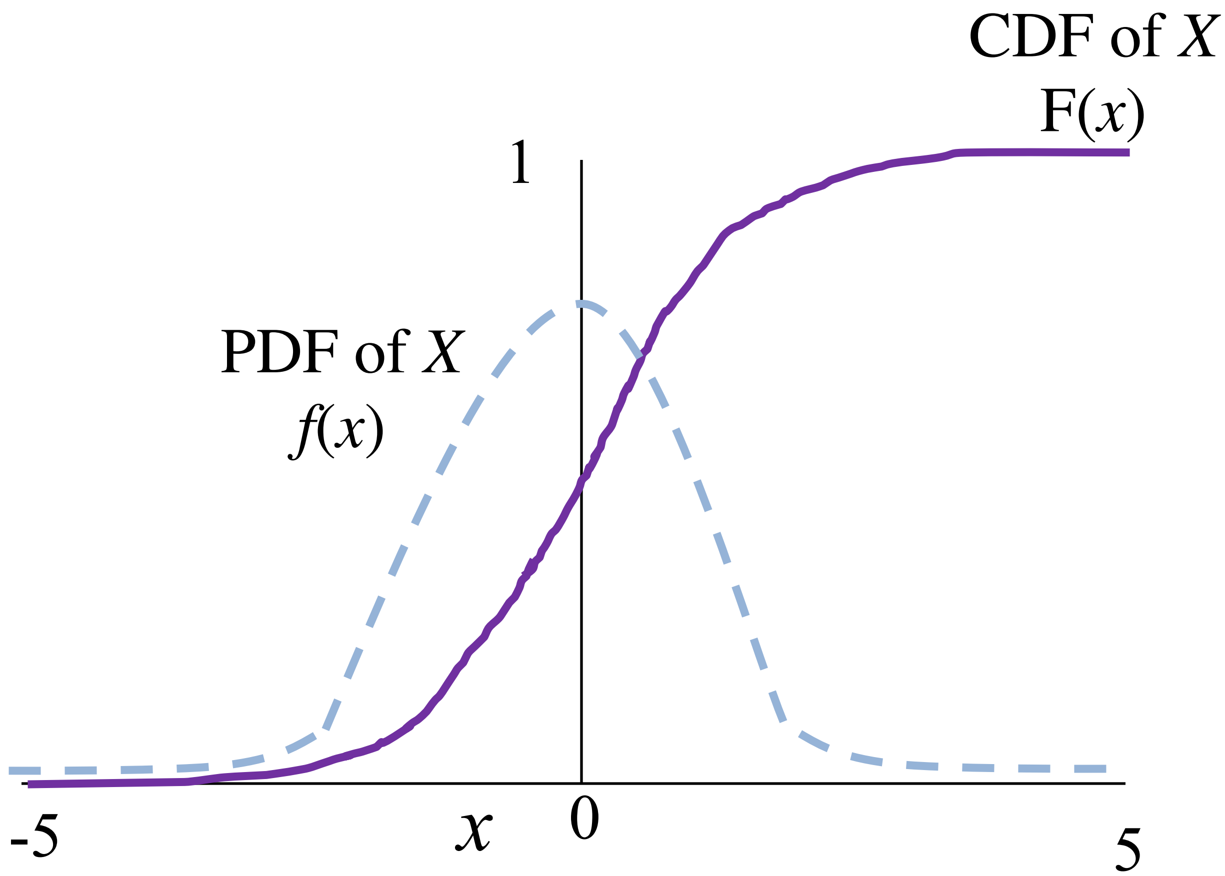
Notation

$p(a)$ or $p_X(a)$ Probability Mass Function (discrete) $P(X = a)$

$f(a)$ or $f_X(a)$ Probability Density Function (continuous) $f(X = a)$

$F(a)$ or $F_X(a)$ Cumulative Distribution Function $P(X \leq a)$

Density vs Cumulative

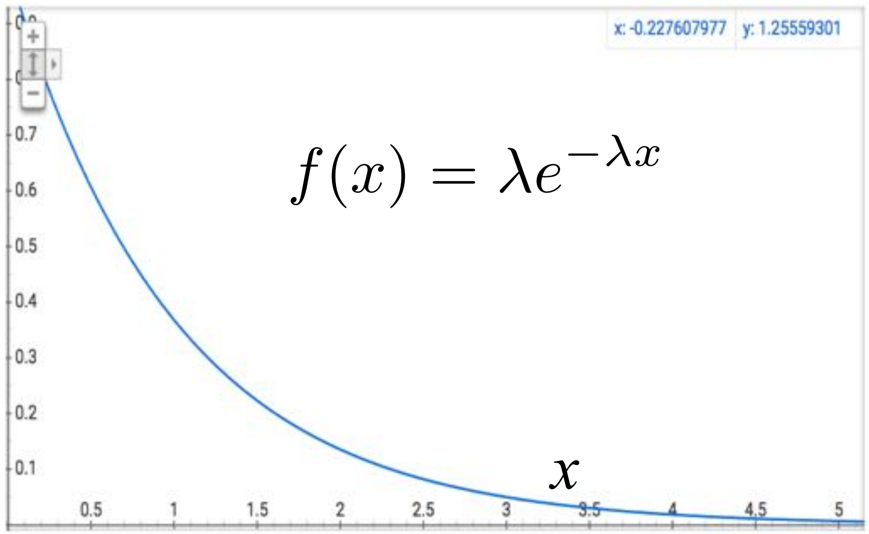


f(x) = derivative of probability

F(x) = P(X < x)

CDF: X ~ Exp(λ = 1)

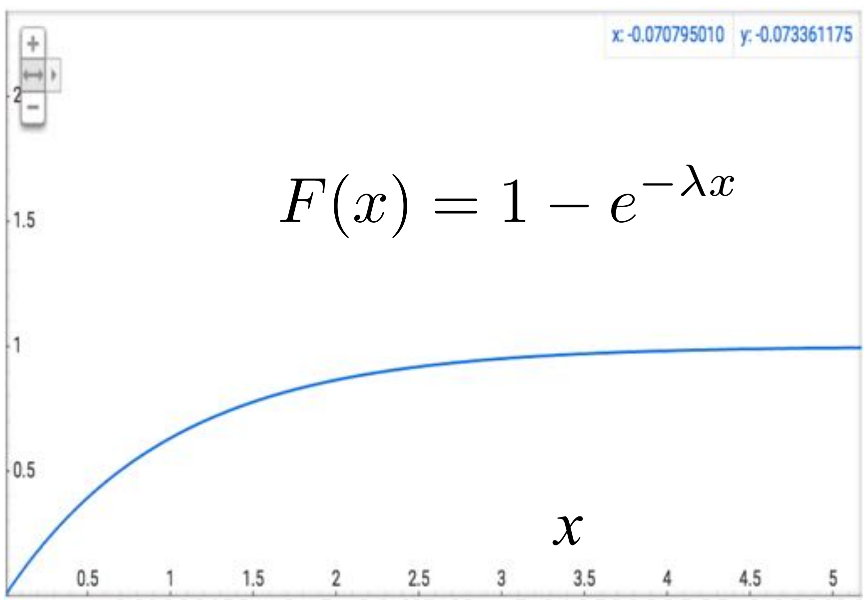
Probability density function



Cumulative density function

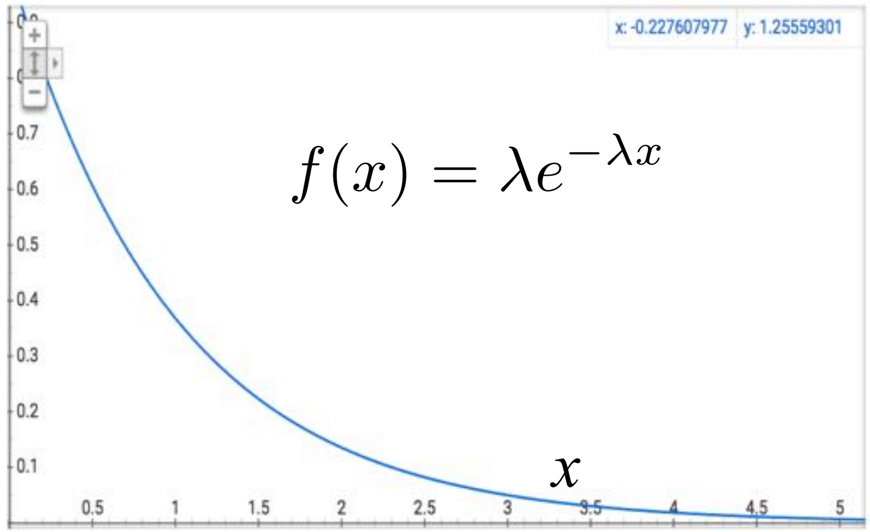
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



CDF: X ~ Exp(λ = 1)

Probability density function

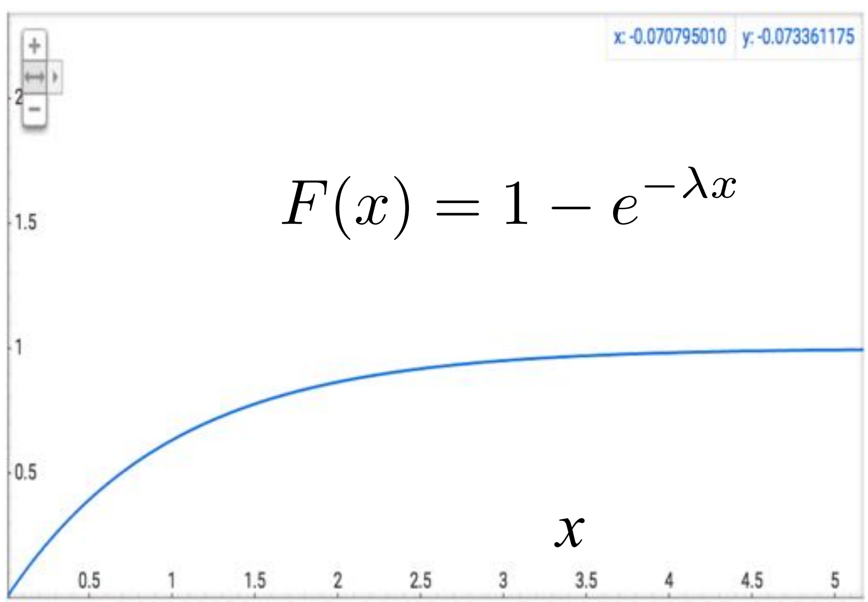


$P(X < 2)$

Cumulative density function

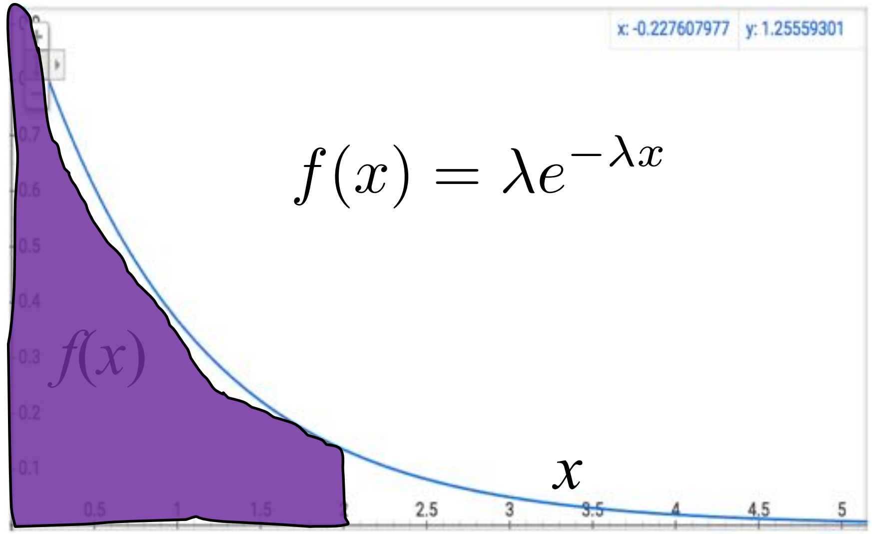
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



CDF: X ~ Exp(λ = 1)

Probability density function



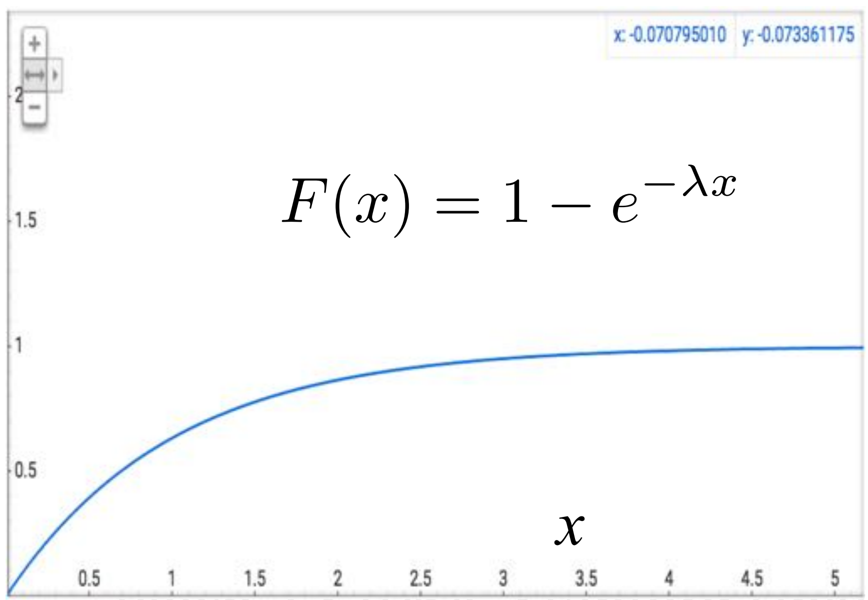
P(X < 2)

$$= \int_{x=-\infty}^{2} f(x)\, dx$$

Cumulative density function

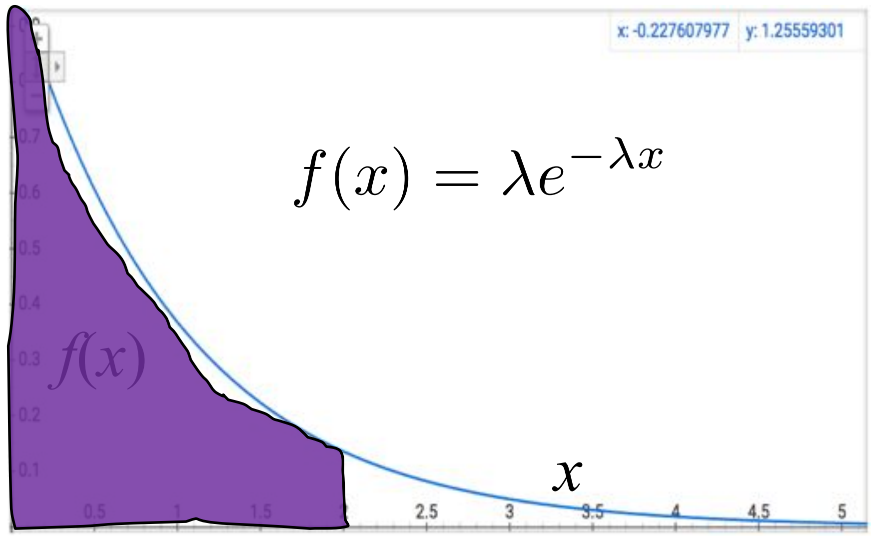
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



CDF: X ~ Exp(λ = 1)

Probability density function



$$f(x) = \lambda e^{-\lambda x}$$

P(X < 2)

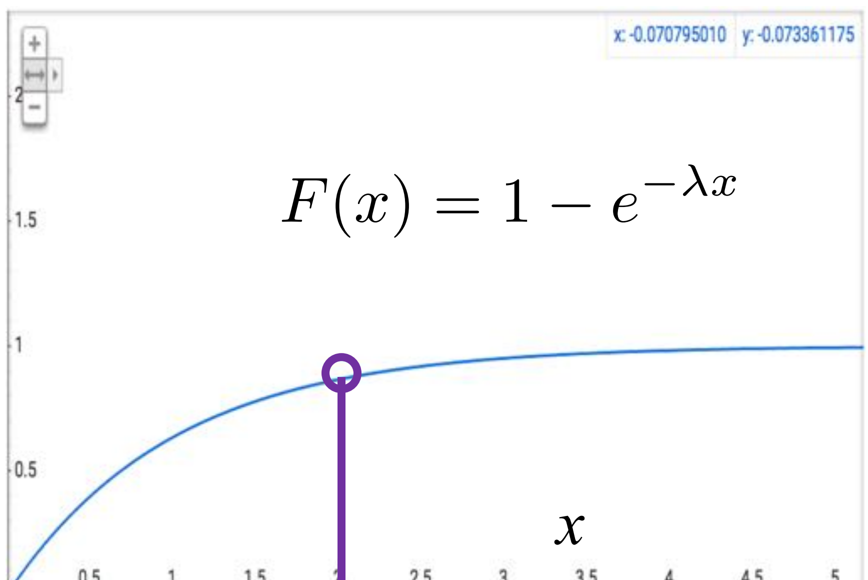
$$= \int_{x=-\infty}^{2} f(x)\, dx$$

or

Cumulative density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



$$F(x) = 1 - e^{-\lambda x}$$

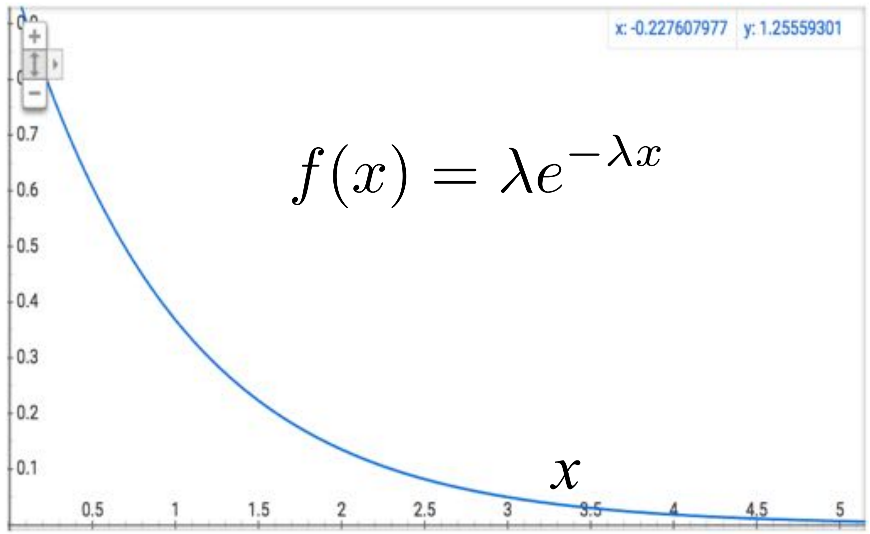
$$= F(2)$$

$$= 1 - e^{-2}$$

$$\approx 0.84$$

CDF: X ~ Exp(λ = 1)

Probability density function

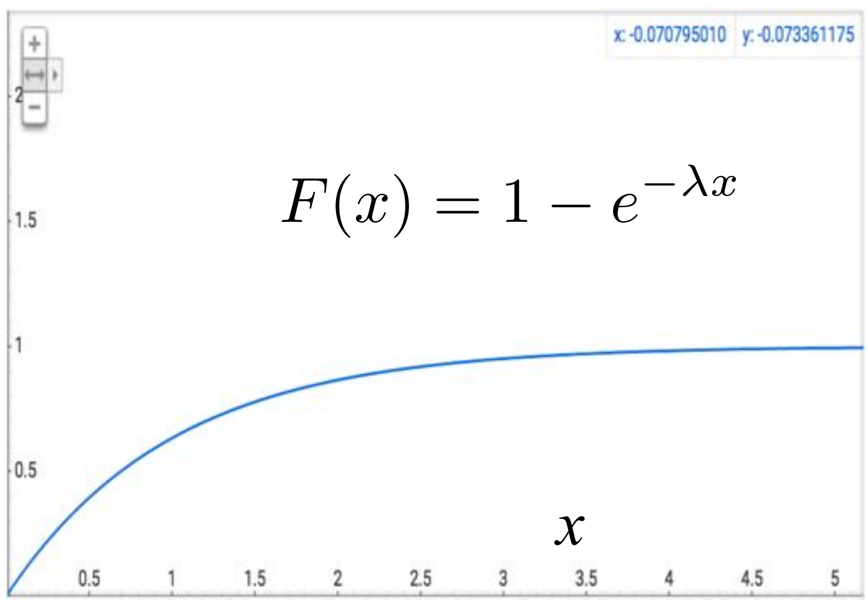


P(X > 1)

Cumulative density function

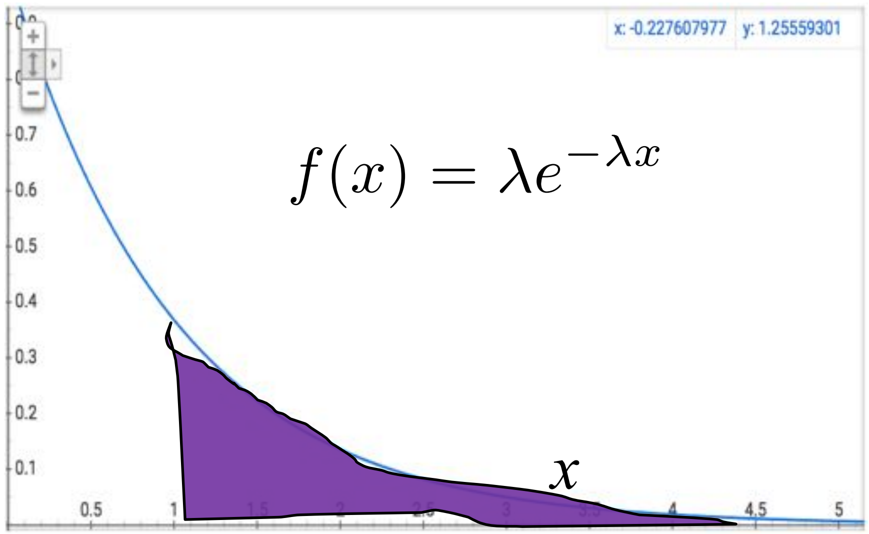
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



CDF: X ~ Exp(λ = 1)

Probability density function



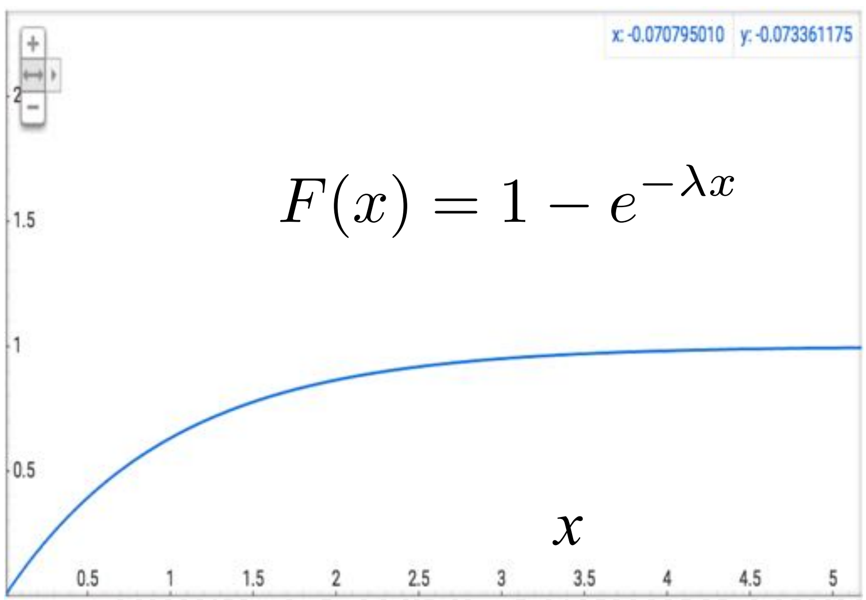
P(X > 1)

$$= \int_{x=1}^{\infty} f(x)\, dx$$

Cumulative density function

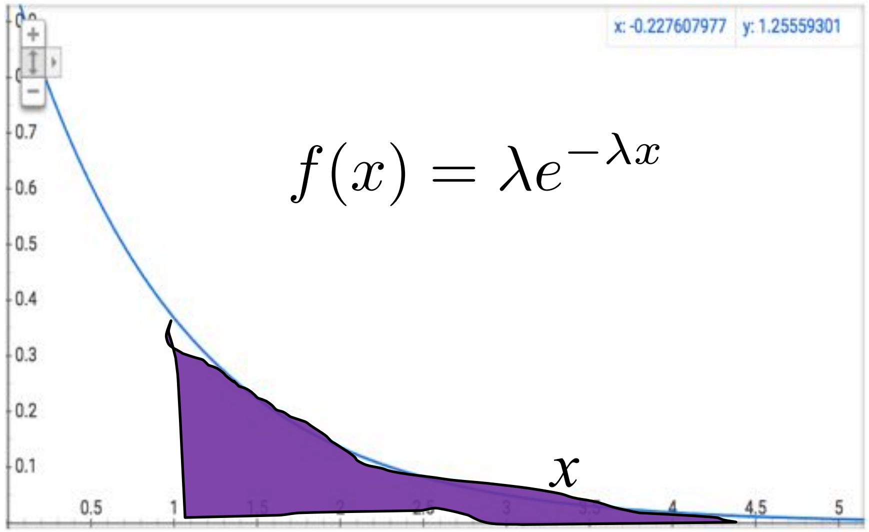
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



CDF: X ~ Exp(λ = 1)

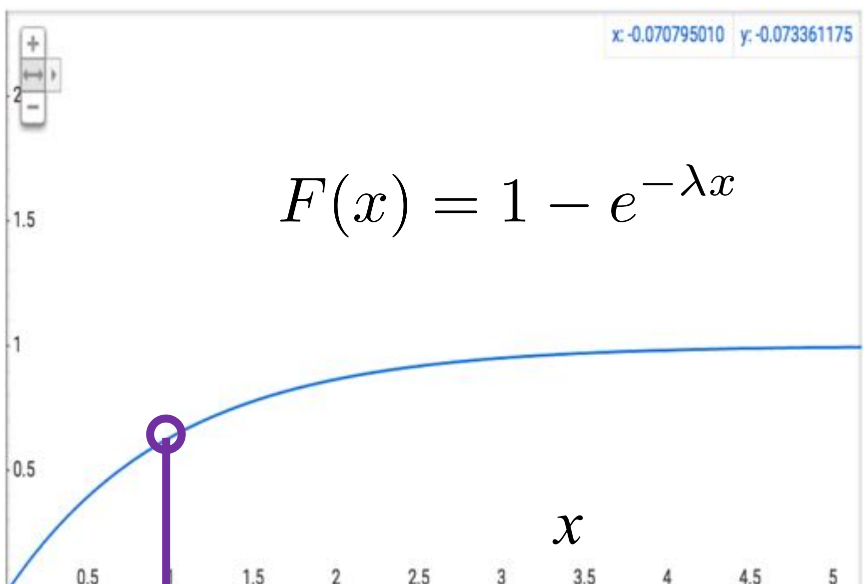
Probability density function



Cumulative density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



P(X > 1)

$$= \int_{x=1}^{\infty} f(x)\, dx$$

or

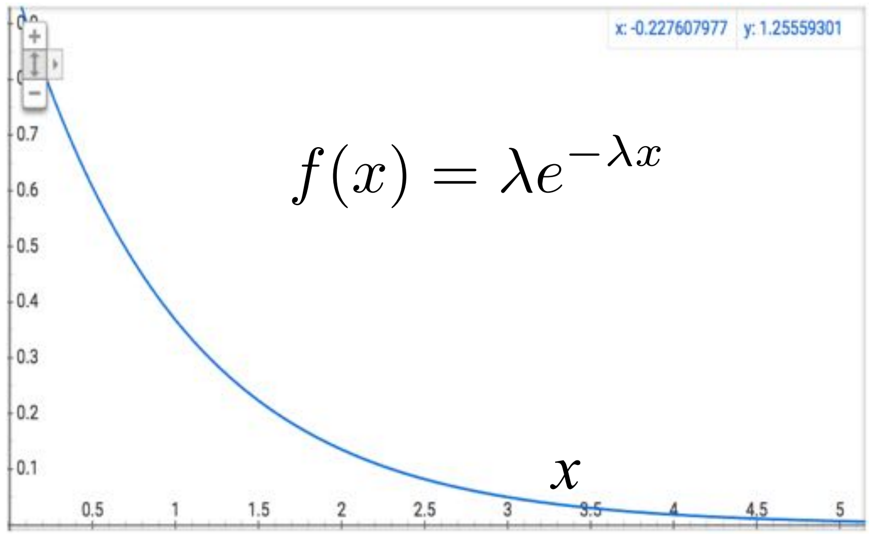
$$= 1 - F(1)$$

$$= e^{-1}$$

$$\approx 0.37$$

CDF: X ~ Exp(λ = 1)

Probability density function

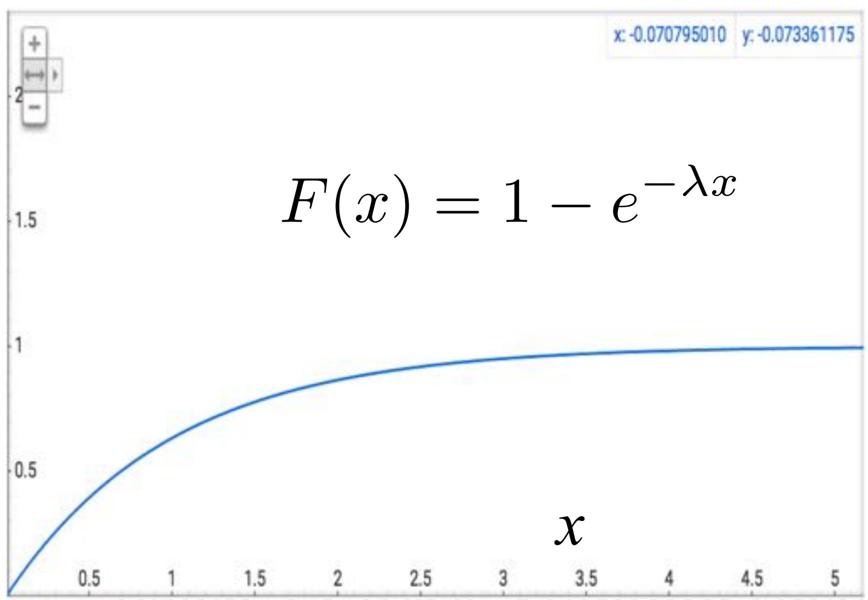


P(1 < X < 2)

Cumulative density function

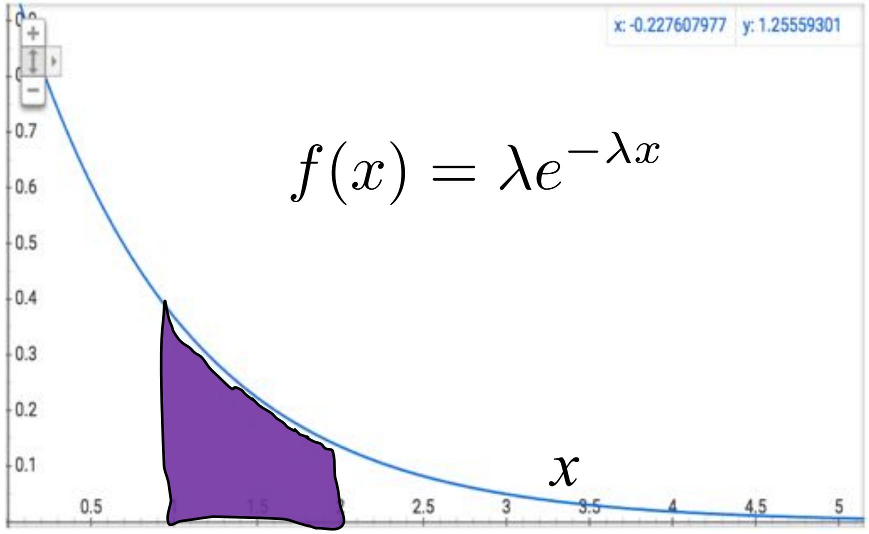
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



CDF: X ~ Exp(λ = 1)

Probability density function



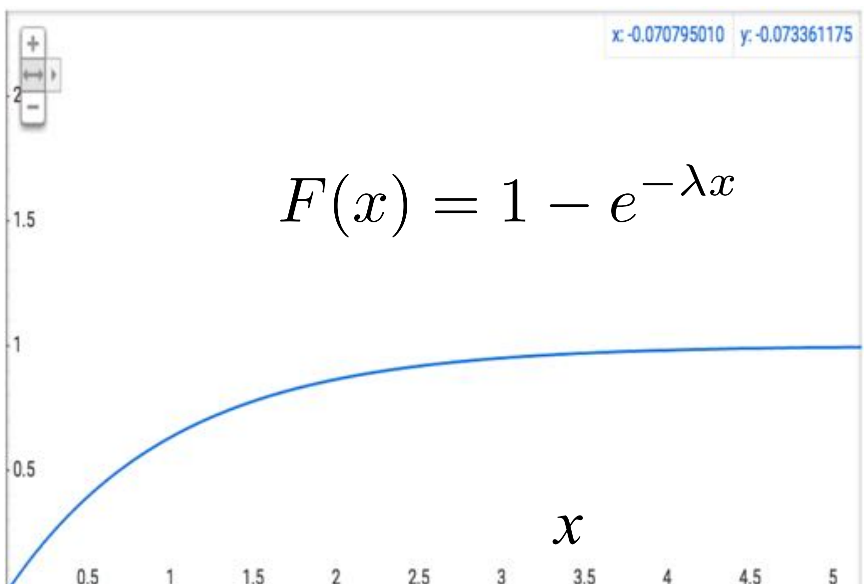
$$P(1 < X < 2)$$

$$= \int_{x=1}^{2} f(x)\, dx$$

Cumulative density function

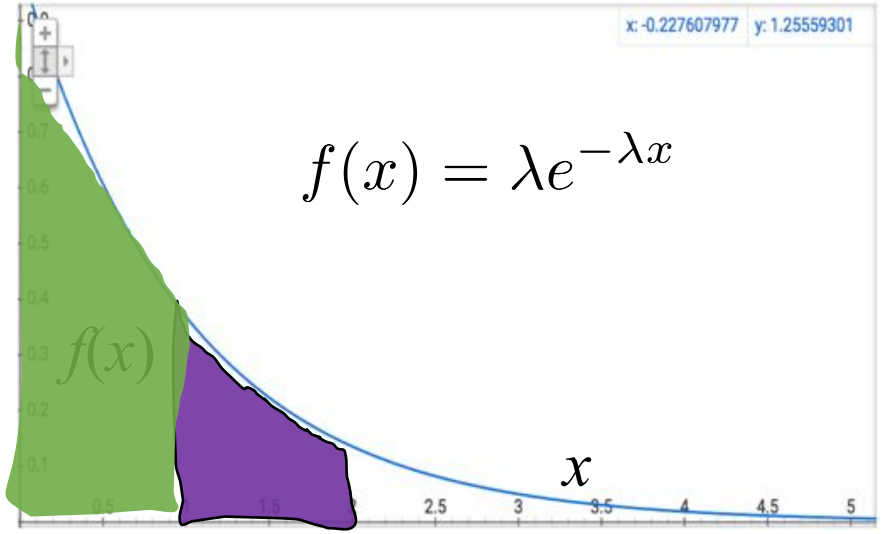
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



CDF: X ~ Exp($\lambda = 1$)

Probability density function



$$P(1 < X < 2)$$

$$= \int_{x=1}^{2} f(x)\, dx$$

or

$$= F(2) - F(1)$$

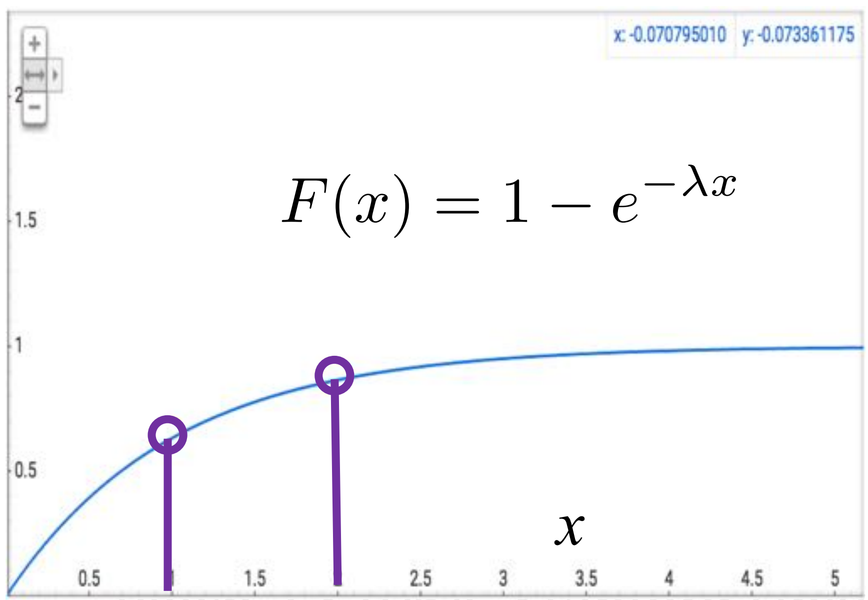
$$= (1 - e^{-2}) - (1 - e^{-1})$$

$$\approx 0.23$$

Cumulative density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$

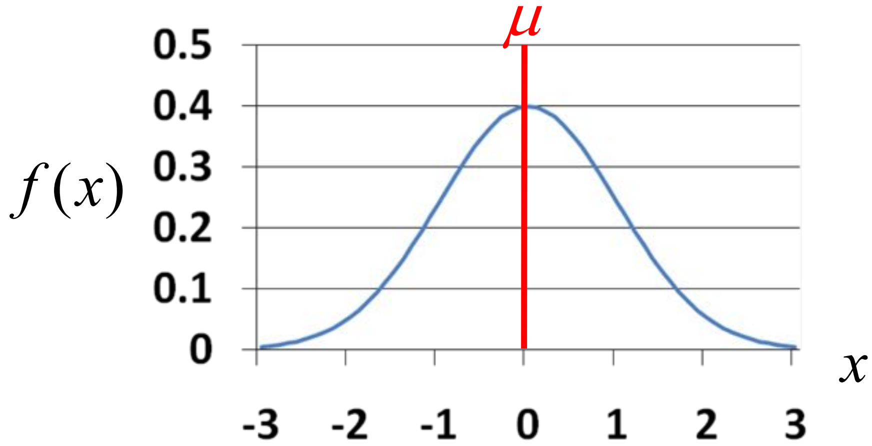


Big Day

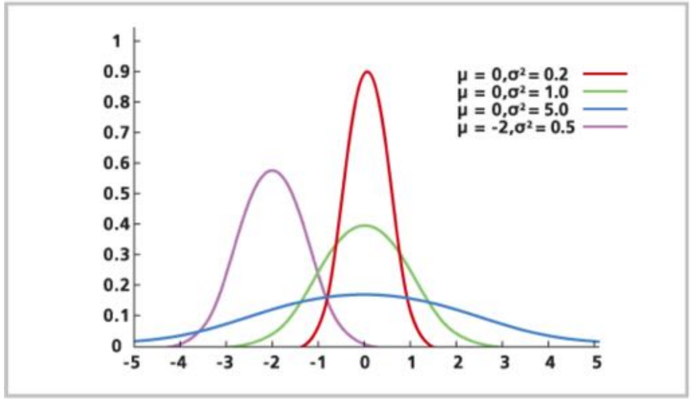
The Normal Distribution

- X is a **Normal Random Variable**: $X \sim N(\mu, \sigma^2)$
  - Probability Density Function (PDF):

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{where} \quad -\infty < x < \infty$$



  - $E[X] = \mu$
  - $Var(X) = \sigma^2$
  - Also called "Gaussian"
  - Note: $f(x)$ is symmetric about μ



Why use the normal?

- Common for natural phenomena: heights, weights, etc.
- Often results from the sum of multiple variables
- Most noise is Normal.
- Sample means are distributed normally.

Or that is what they want you to believe

But I Encourage you to be Critical

These are log-normal

- Common for natural phenomena: heights, weights, etc.

Only if they are equally weighted

- Often results from the sum of multiple variables

Most noise is assumed normal

- Most noise is Normal.

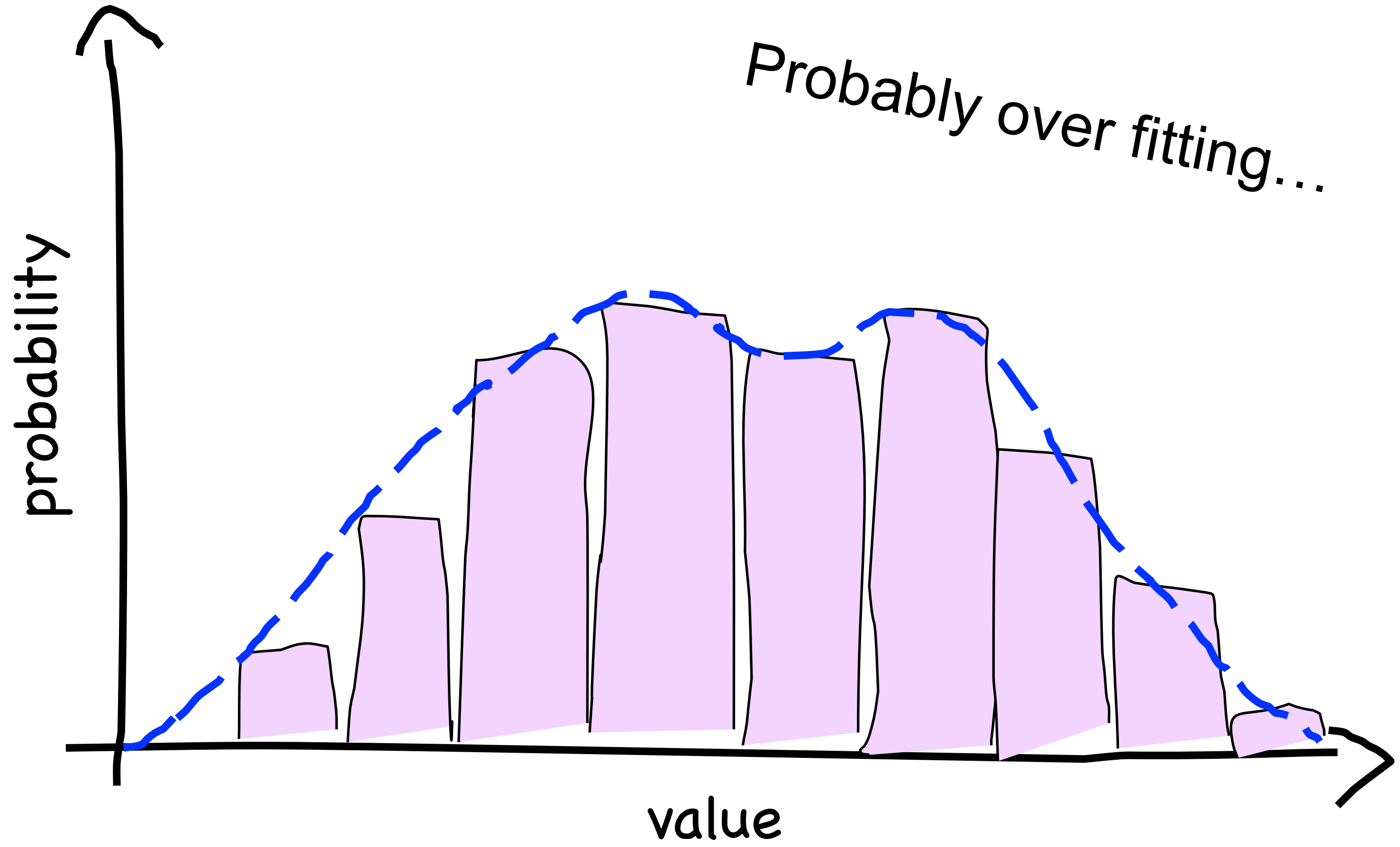
- Sample means are distributed normally.

It is the most important distribution

Because of a deeper truth…

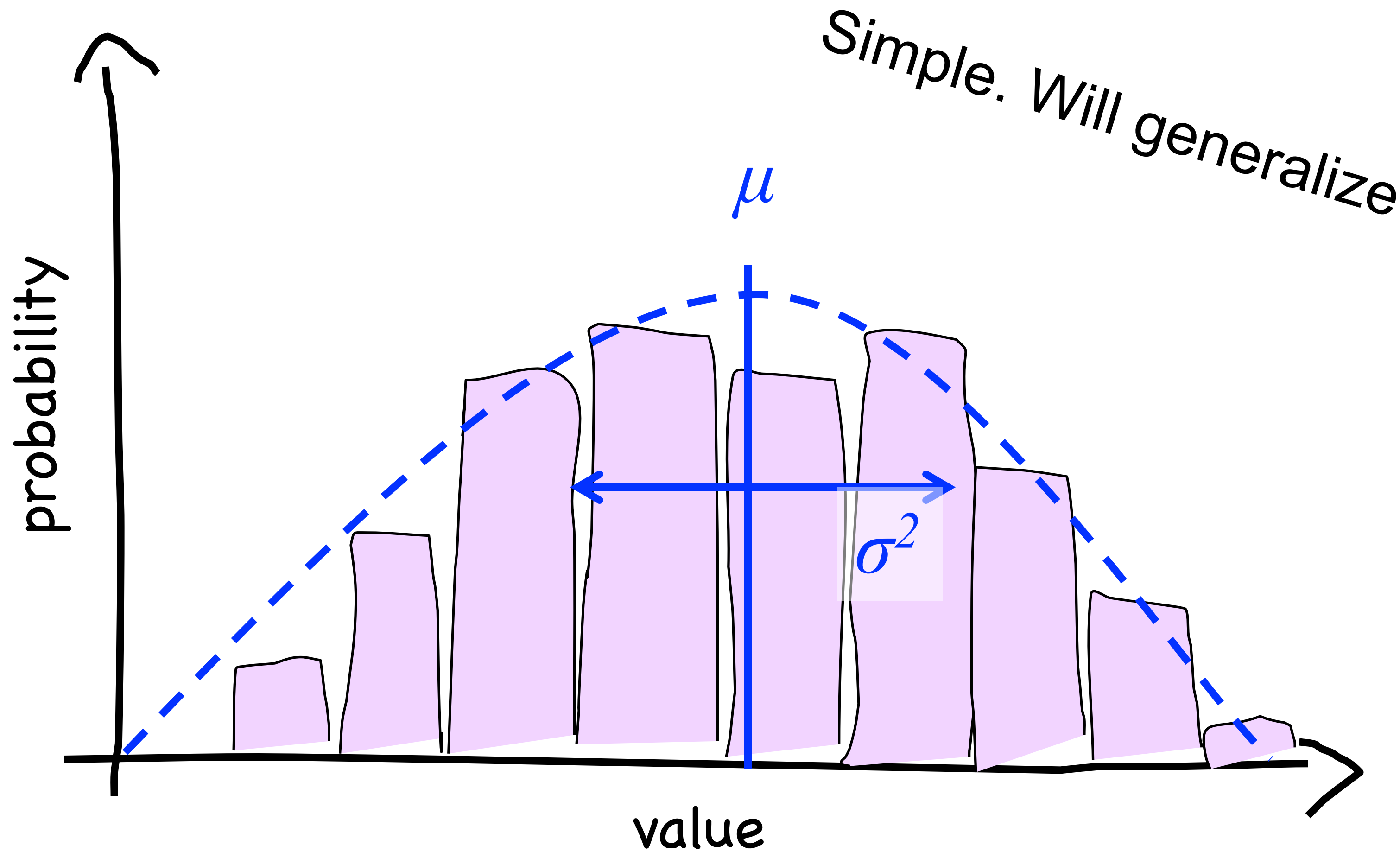
“The simplest explanation is usually the best one”

Complexity is Tempting



* That describes the training data, but will it generalize?

Simplicity is Humble



* A Gaussian maximizes entropy for a given mean and variance

Carl Friedrich Gauss

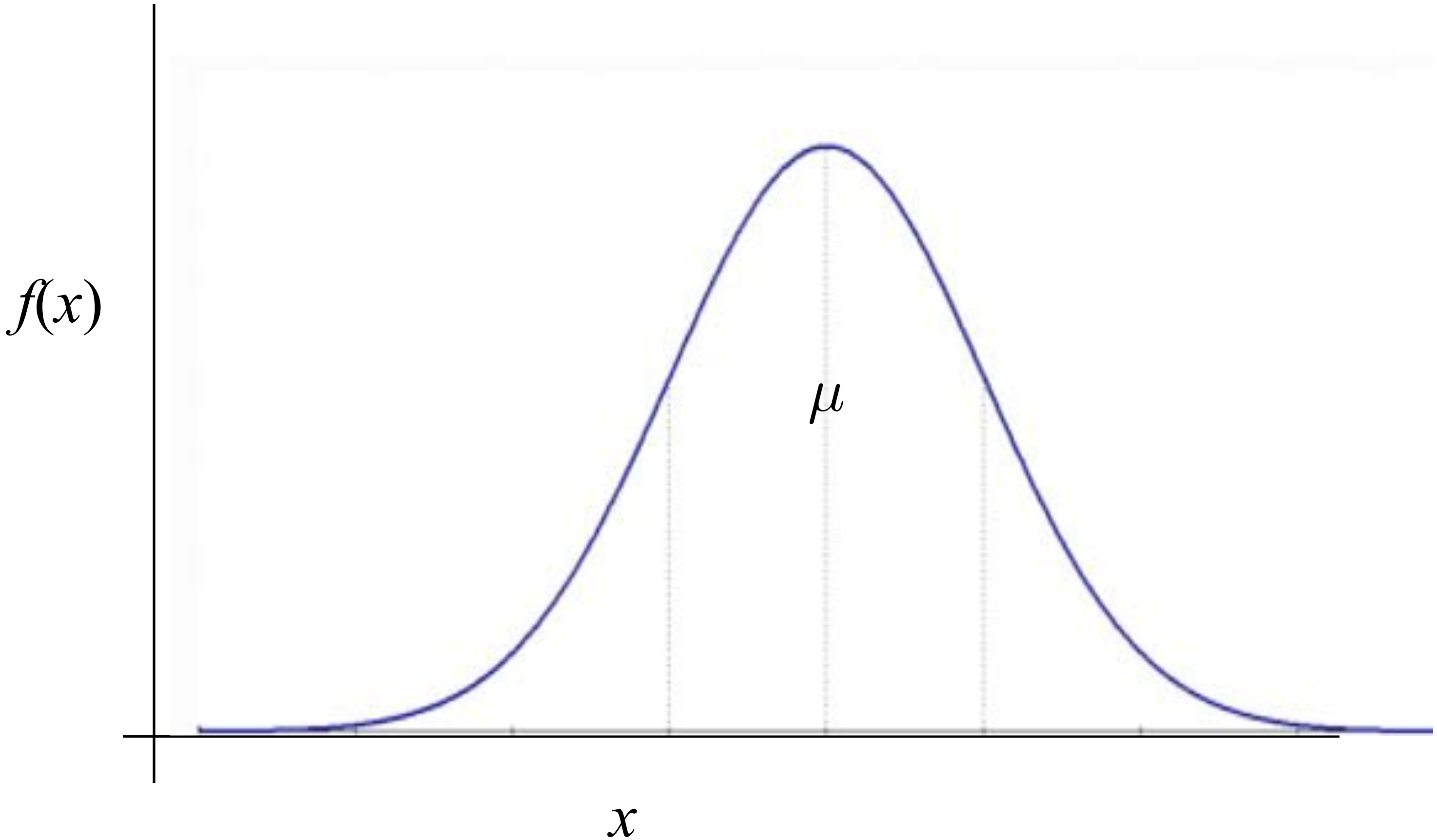
- Carl Friedrich Gauss (1777-1855) was a remarkably influential German mathematician

- Started doing groundbreaking math as teenager
  - Did not invent Normal distribution, but popularized it
- He looked like Martin Sheen
  - Who is, of course, Charlie Sheen's father

Probability Density Function

$$\mathcal{N}(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



Anatomy of a beautiful equation

$\mathcal{N}(\mu, \sigma^2)$

"exponential"

the distance to the mean

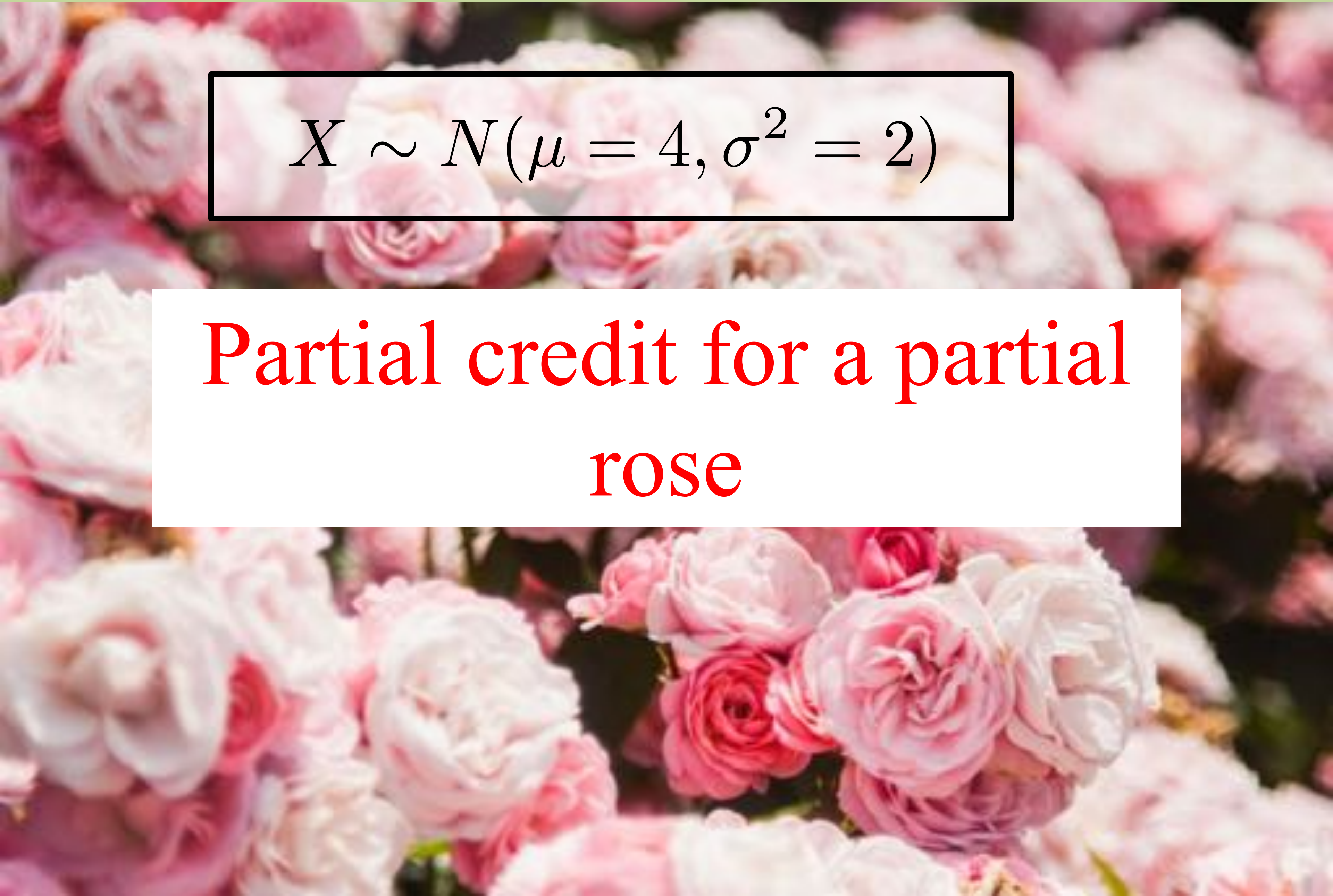
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

probability density at x

a constant

sigma shows up twice

Flowers on a Rose Bush

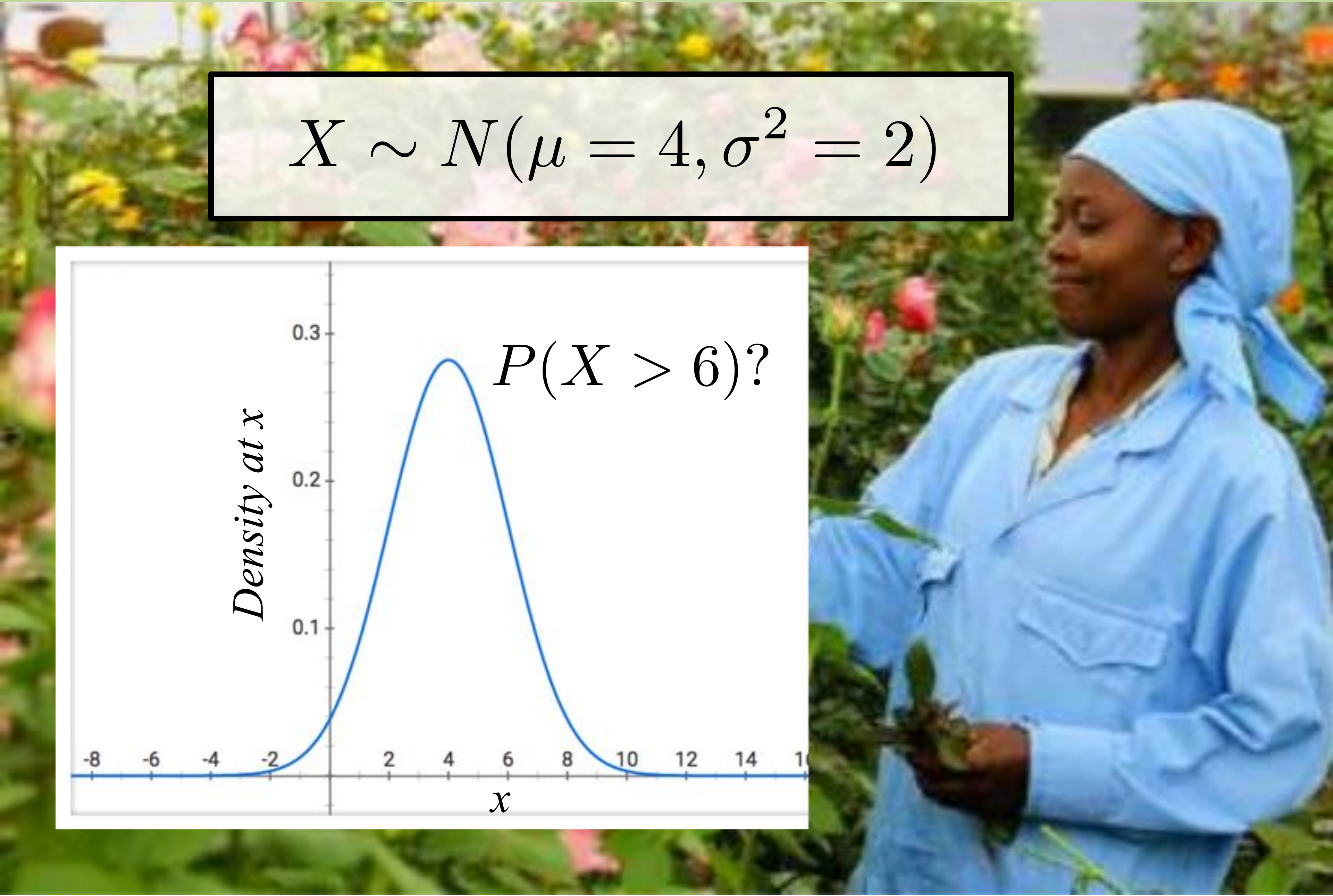


$$X \sim N(\mu = 4, \sigma^2 = 2)$$

Partial credit for a partial rose

Scientist from Kenya

$$X \sim N(\mu = 4, \sigma^2 = 2)$$



Let's try and integrate it!

$$P(a \leq X \leq b) =$$

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

* Call me if you find an equation for this

No closed form for the integral

No closed form for $F(x)$

Spoiler Alert

$\mathcal{N}(\mu, \sigma^2)$

A function that has been solved for numerically

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function of any normal

* We are going to spend the next few slides getting here

Linear Transform of Normal is Normal

Let $X \sim \mathcal{N}(\mu, \sigma^2)$

If $Y = aX + b$ then Y is also Normal

$$\begin{aligned} E[Y] &= E[aX + b] \\ &= aE[X] + b \\ &= a\mu + b \end{aligned}$$

$$\begin{aligned} Var(Y) &= Var(aX + b) \\ &= a^2 Var(X) \\ &= a^2\sigma^2 \end{aligned}$$

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

Special Linear Transform

If $Y = aX + b$ then Y is also Normal

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

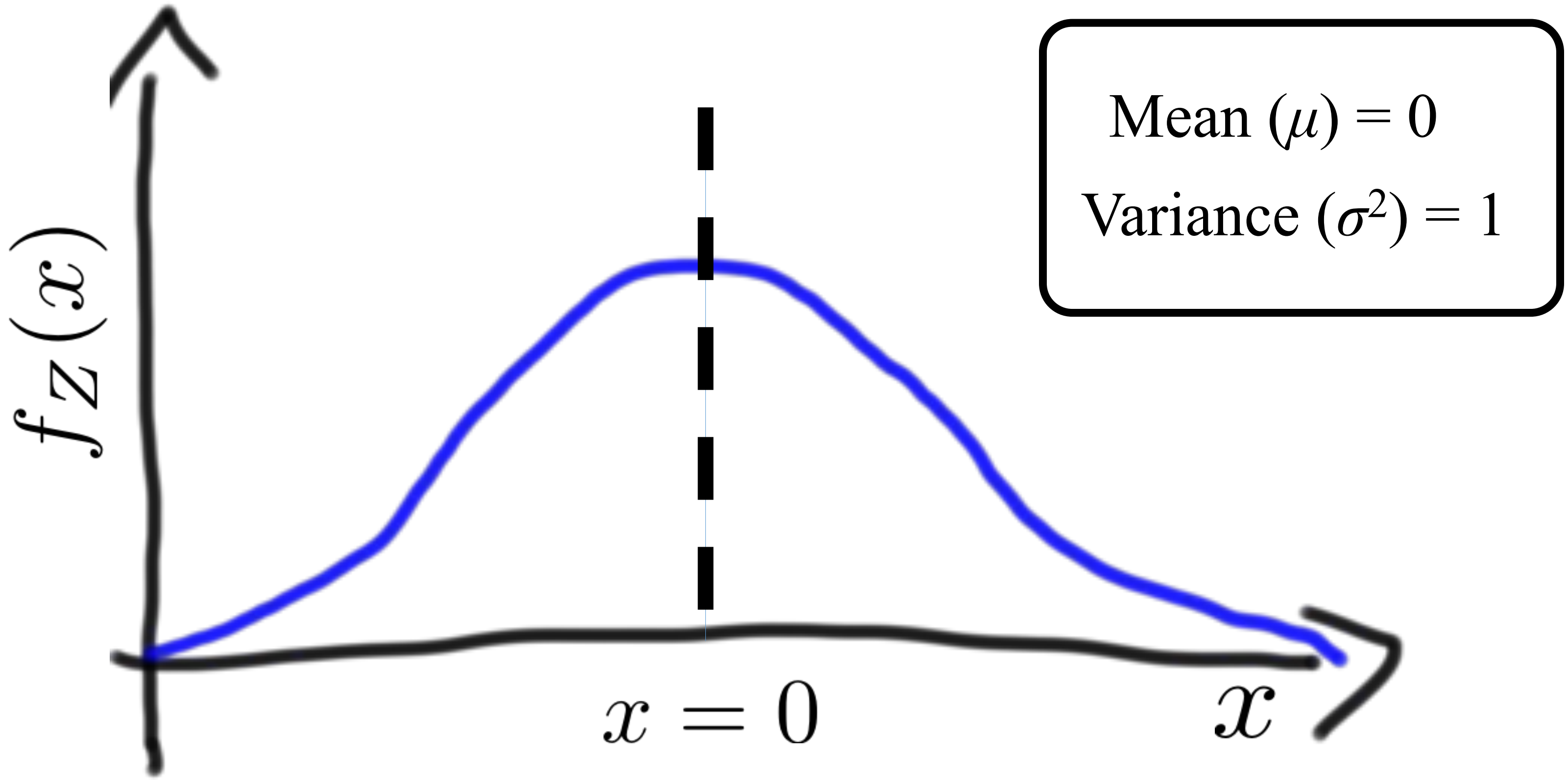
There is a special case of linear transform for any X:

$$Z = \frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma} \qquad a = \frac{1}{\sigma} \quad b = -\frac{\mu}{\sigma}$$

$$\begin{aligned} Z &\sim \mathcal{N}(a\mu + b, a^2\sigma^2) \\ &\sim \mathcal{N}(\frac{\mu}{\sigma} - \frac{\mu}{\sigma}, \frac{\sigma^2}{\sigma^2}) \\ &\sim \mathcal{N}(0, 1) \end{aligned}$$

The Standard Normal

$$Z \sim N(\mu = 0, \sigma^2 = 1)$$

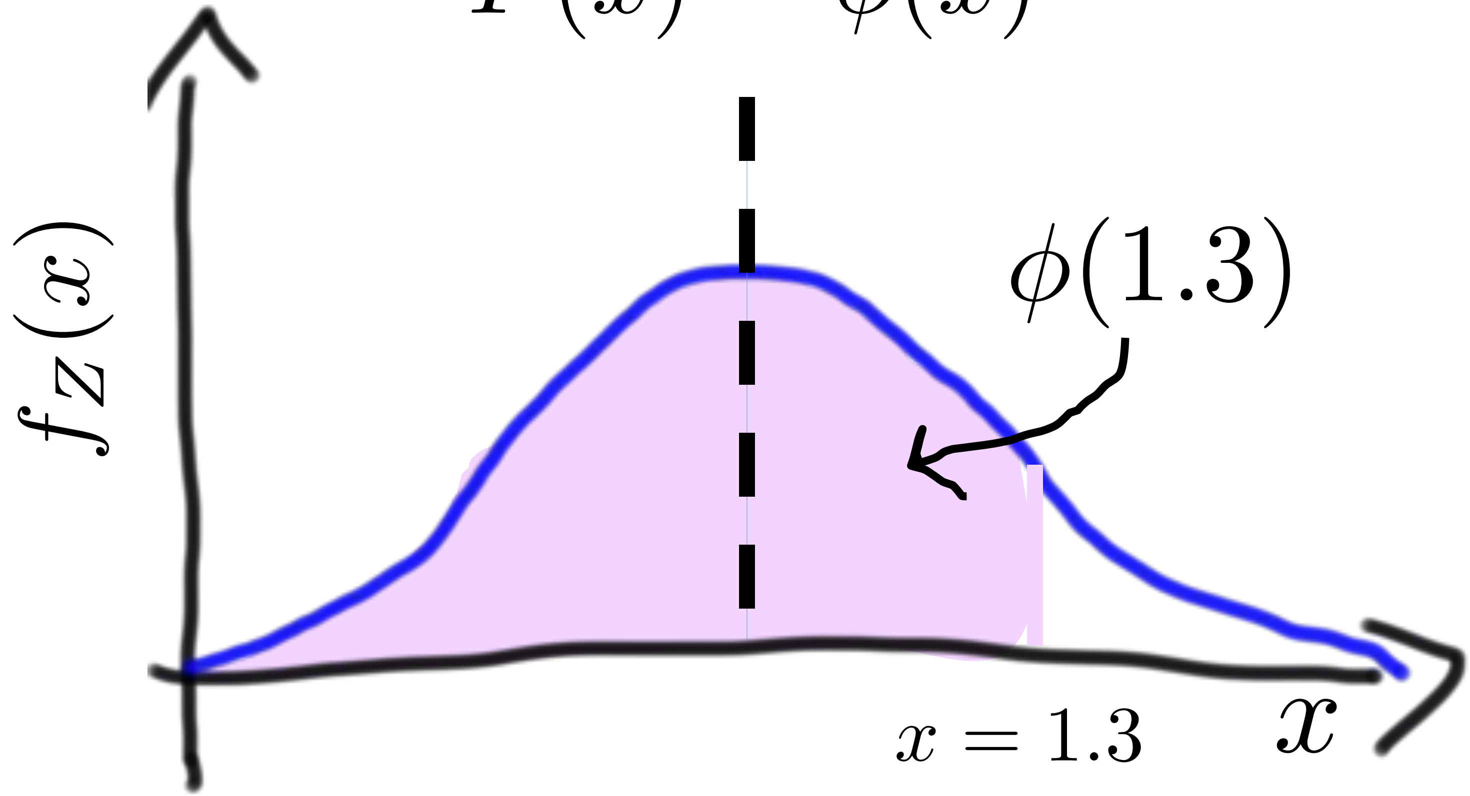


*This is the probability density function for the standard normal

Phi

$$Z \sim N(\mu = 0, \sigma^2 = 1)$$

$$F(x) = \phi(x)$$

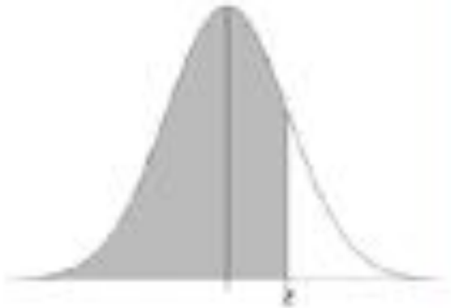


*This is the probability density function for the standard normal

Using Table of ϕ

Standard Normal Cumulative Probability Table

$\Phi(1.31) = 0.7054$

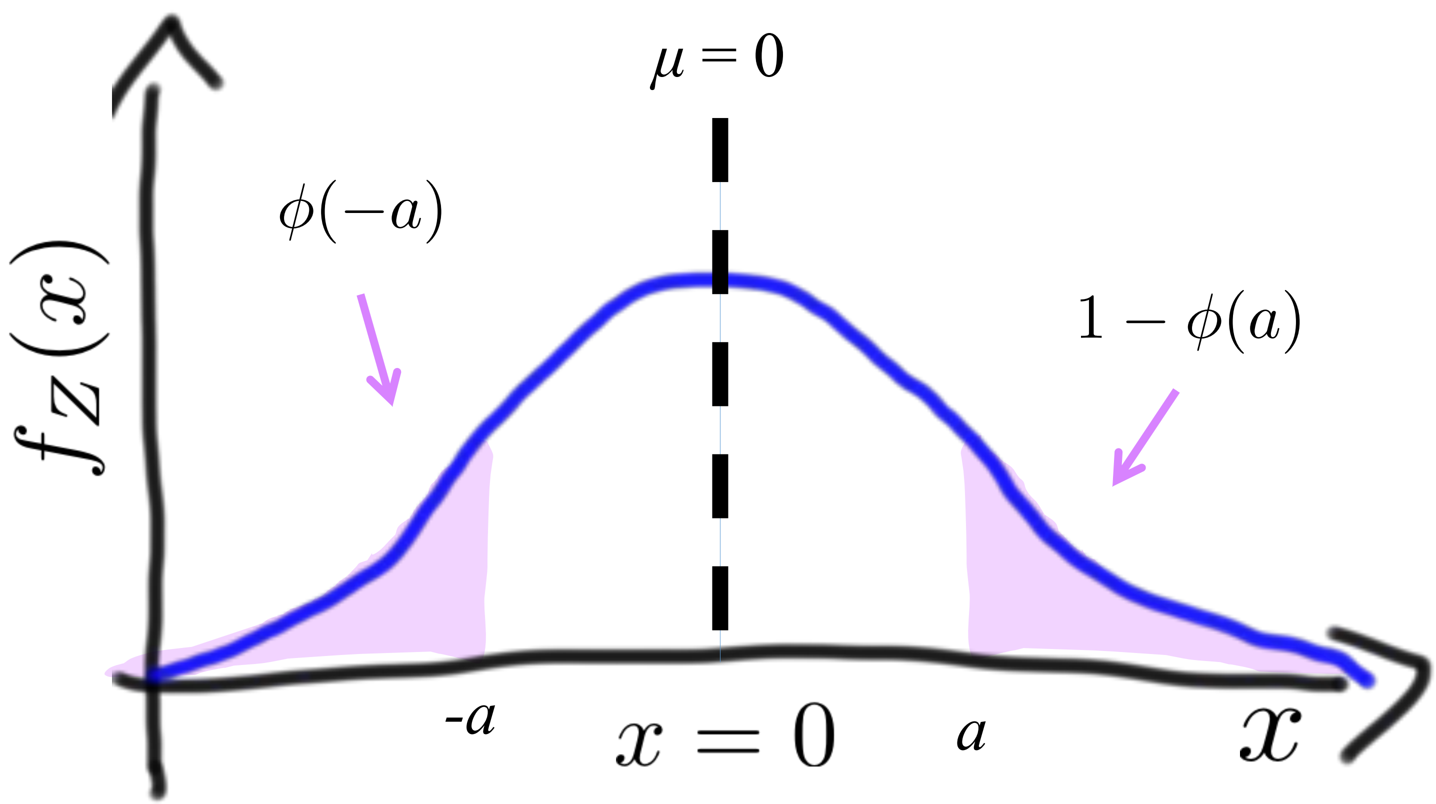


Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

Symmetry of Phi

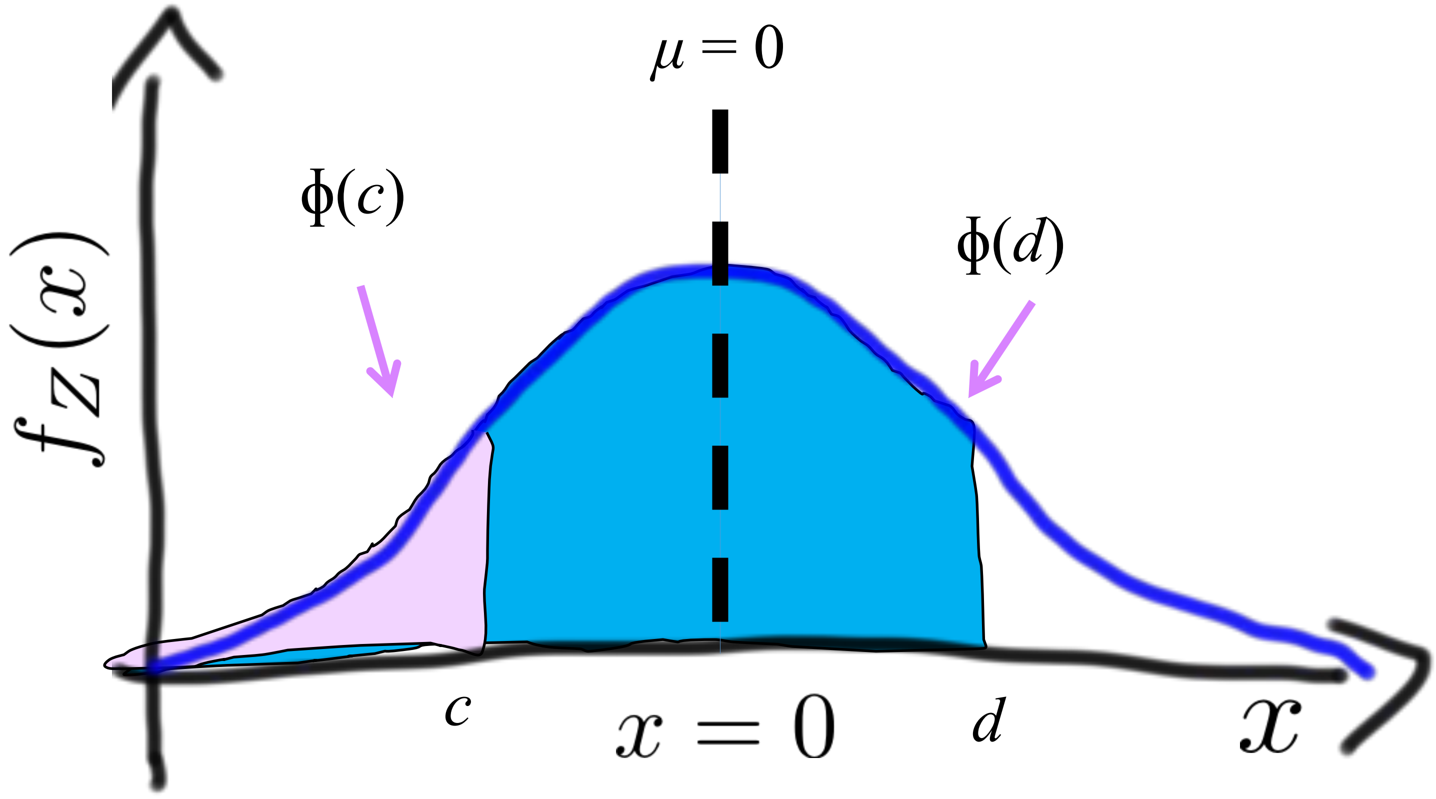
$$\phi(a) = 1 - \phi(a)$$



*This is the probability density function for the standard normal

Interval of Phi

$$P(c < Z < d) = \phi(d) - \phi(c)$$



Compute F(x) via Transform

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ $\quad Z = \dfrac{X - \mu}{\sigma}$

Use Z to compute $F(x)$

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(X - \mu \leq x - \mu) \\
&= P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) \\
&= P\left(Z \leq \frac{x - \mu}{\sigma}\right)
\end{aligned}
$$

$$= \Phi\left(\frac{x - \mu}{\sigma}\right)$$

For normal distribution, F(x) is computed using the phi transform.

And here we are

$\mathcal{N}(\mu, \sigma^2)$

CDF of Standard Normal: A function that has been solved for numerically

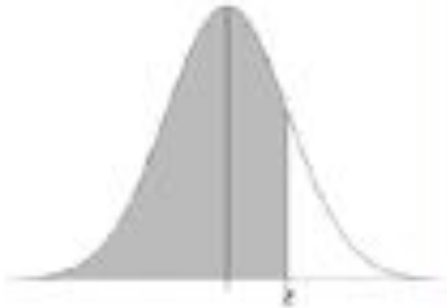
$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function (CDF) of any normal

Table of $\Phi(z)$ values in textbook, p. 201 and handout

Using Table of ϕ

Standard Normal Cumulative Probability Table



$\Phi(0.54) = 0.7054$

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

Table is kinda old school

Using Programming Library

Every modern programming language has a normal library

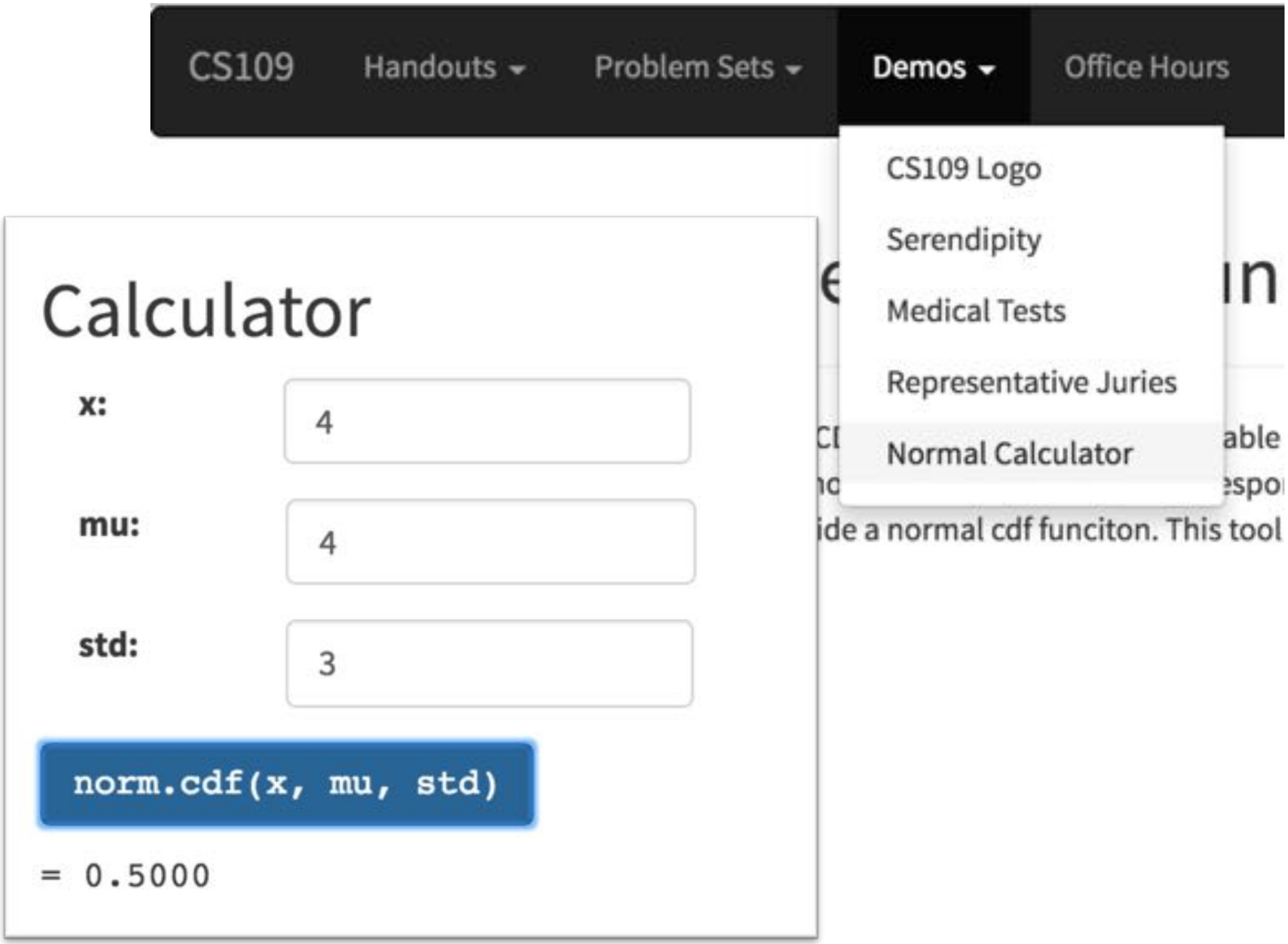
```
norm.cdf(x, mean, std)
```

$$= P(X < x) \text{ where } X \sim \mathcal{N}(\mu, \sigma^2)$$

$$= \Phi\left(\frac{x - \mu}{\sigma}\right)$$

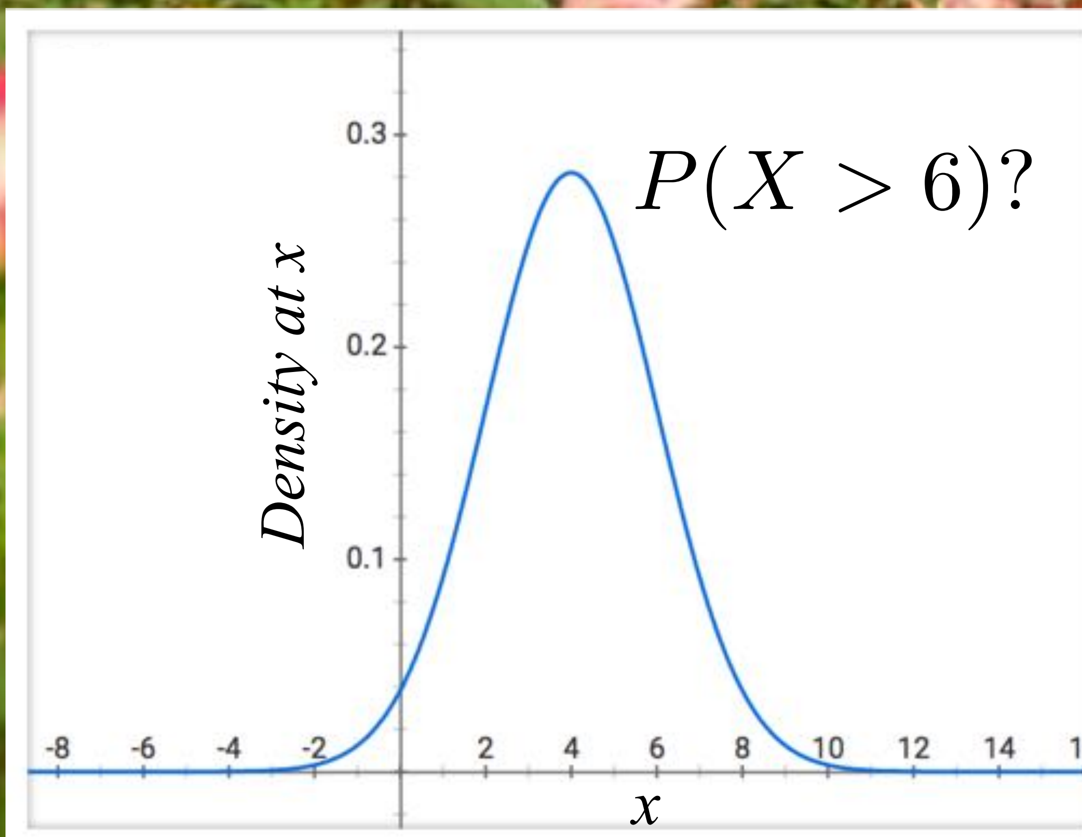
* This is from Python's scipy library

I made one for you



Flowers on a Rose Bush

$$X \sim N(\mu = 4, \sigma^2 = 2)$$



$$P(X > 6)?$$

Flowers on a Rose Bush

flowers on a rose bush

$$X \sim N(\mu = 4, \sigma^2 = 2)$$

$$P(X > 6)?$$

$$P(X > 6) = 1 - F_X(6)$$

$$= 1 - \Phi\left(\frac{6-\mu}{\sigma}\right)$$

$$= 1 - \Phi\left(\frac{6-4}{\sqrt{2}}\right)$$

$$\approx 1 - \Phi\left(1.414\right)$$

$$\approx 0.079$$

For any normal:

$$F_X(x) = \phi\left(\frac{x-\mu}{\sigma}\right)$$

Get Your Gaussian On

$$X \sim N(\mu = 3, \sigma^2 = 16) \qquad \mu = 3 \qquad \sigma = 4$$

$$\begin{aligned} P(X > 0) &= 1 - F_X(0) = 1 - \Phi\left(\frac{0-\mu}{\sigma}\right) \\ &= 1 - \Phi\left(\frac{-3}{4}\right) = 1 - \left(1 - \Phi\left(\frac{3}{4}\right)\right) \\ &= \Phi\left(\frac{3}{4}\right) = 0.7734 \end{aligned}$$

$$\begin{aligned} P(2 < X < 5) &= F_X(5) - F_X(2) = \Phi\left(\frac{5-\mu}{\sigma}\right) - \Phi\left(\frac{2-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{5-3}{4}\right) - \Phi\left(\frac{2-3}{4}\right) = \Phi\left(\frac{2}{4}\right) - \Phi\left(\frac{-1}{4}\right) = 0.6915 \end{aligned}$$

$$\begin{aligned} P(|X - 3| > 6) &= P(X < -3) + P(X > 9) = F_X(-3) + (1 - F_X(9)) \\ &= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right) = 0.1337 \end{aligned}$$

Noisy Wires

- Send voltage of 2 or -2 on wire (to denote 1 or 0)
  - X = voltage sent
  - R = voltage received = X + Y, where noise Y ~ N(0, 1)
  - Decode R: if (R ≥ 0.5) then 1, else 0
  - What is P(error after decoding | original bit = 1)?

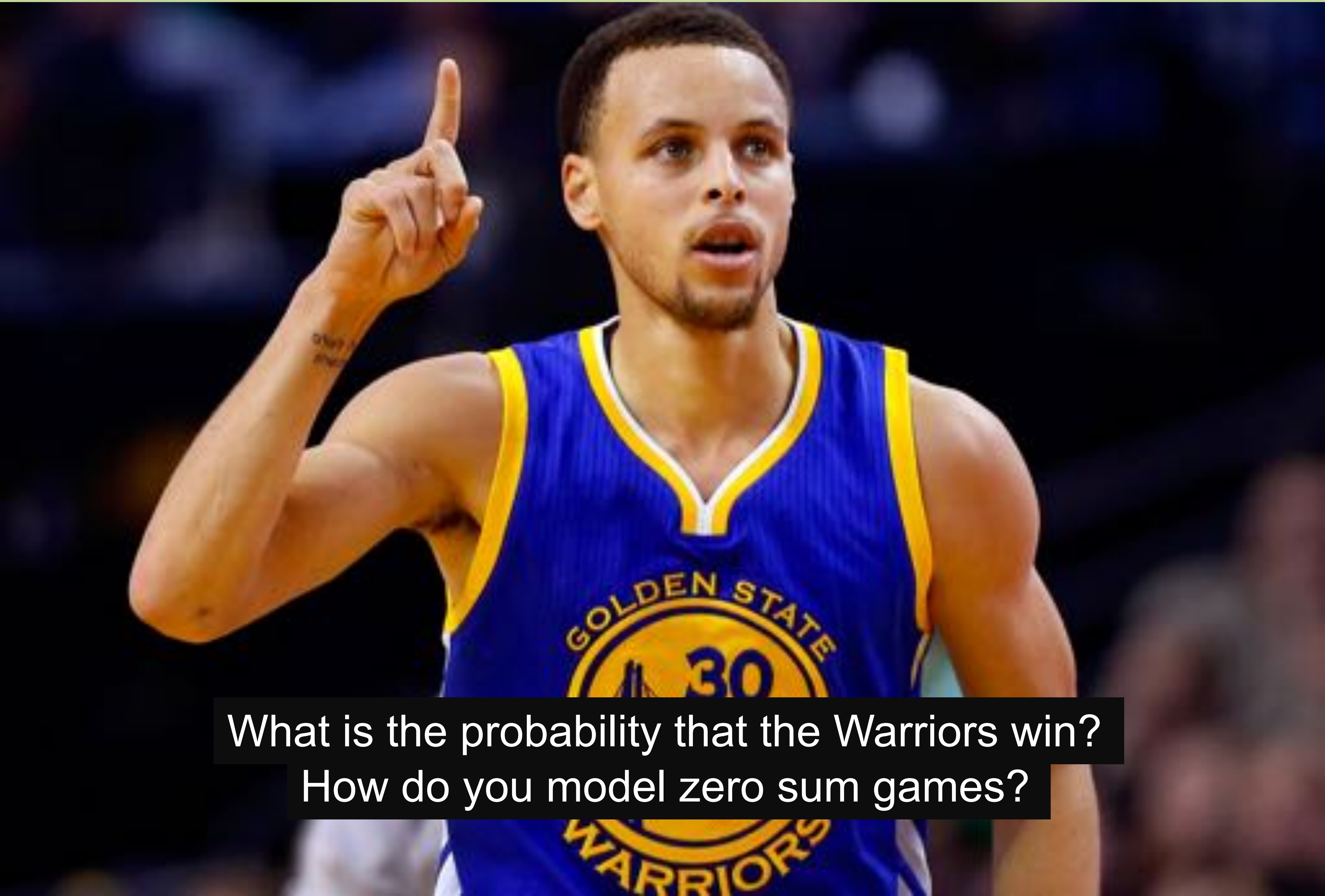
$$P(2+Y<0.5)=P(Y<-1.5)=\Phi(-1.5)=1-\Phi(1.5)\approx 0.0668$$

  - What is P(error after decoding | original bit = 0)?

$$P(-2+Y\geq 0.5)=P(Y\geq 2.5)=1-\Phi(2.5)\approx 0.0062$$

Gaussian for uncertainty

ELO Ratings



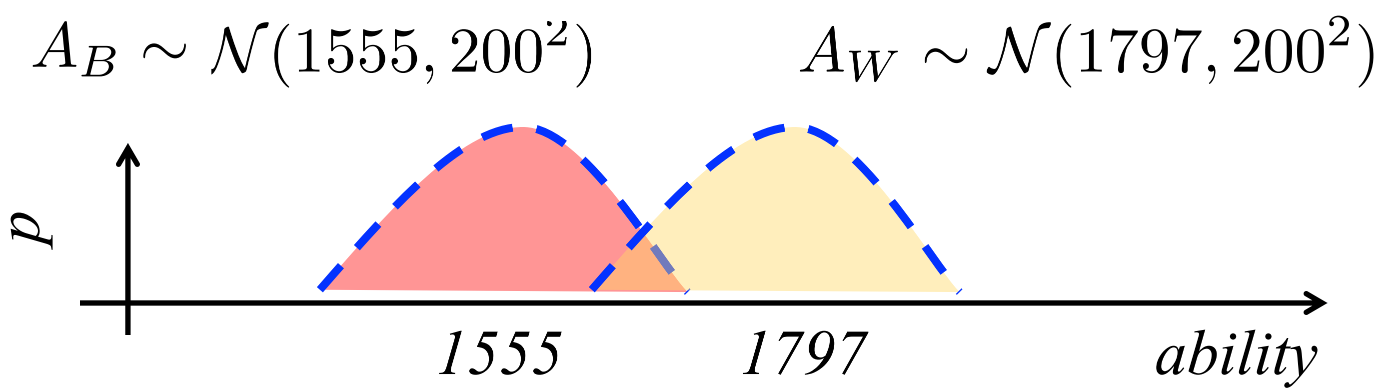
What is the probability that the Warriors win?
How do you model zero sum games?

ELO Ratings

How it works:

- Each team has an "ELO" score *S*, calculated based on their past performance.
- Each game, the team has ability $A \sim N(S, 200^2)$
- The team with the higher sampled ability wins.

Arpad Elo



$$P(\text{Warriors win}) = P(A_W > A_B)$$

ELO Ratings

```
from random import *

WARRIORS_ELO = 1797
OPPONENT_ELO = 1555
VAR = 200 * 200

nSuccess = 0
for i in range(NTRIALS):
    w = gauss(WARRIORS_ELO, VAR)
    b = gauss(OPPONENT_ELO, VAR)
    if w > b:
        nSuccess += 1

print float(nSuccess) / NTRIALS
```

ELO Ratings

How it works:

- Each team has an "ELO" score *S*, calculated based on their past performance.
- Each game, the team has ability $A \sim N(S, 200^2)$
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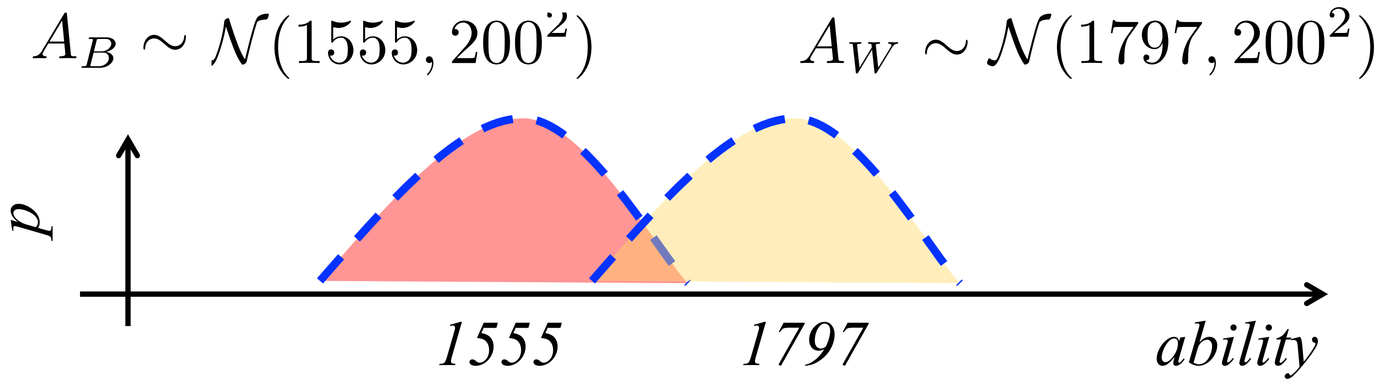
Arpad Elo



$$P(\text{Warriors win}) = P(A_W > A_B)$$
$$\approx 0.87$$

Calculated via sampling

That’s all folks!

If time…

Imagine you are sitting a test…

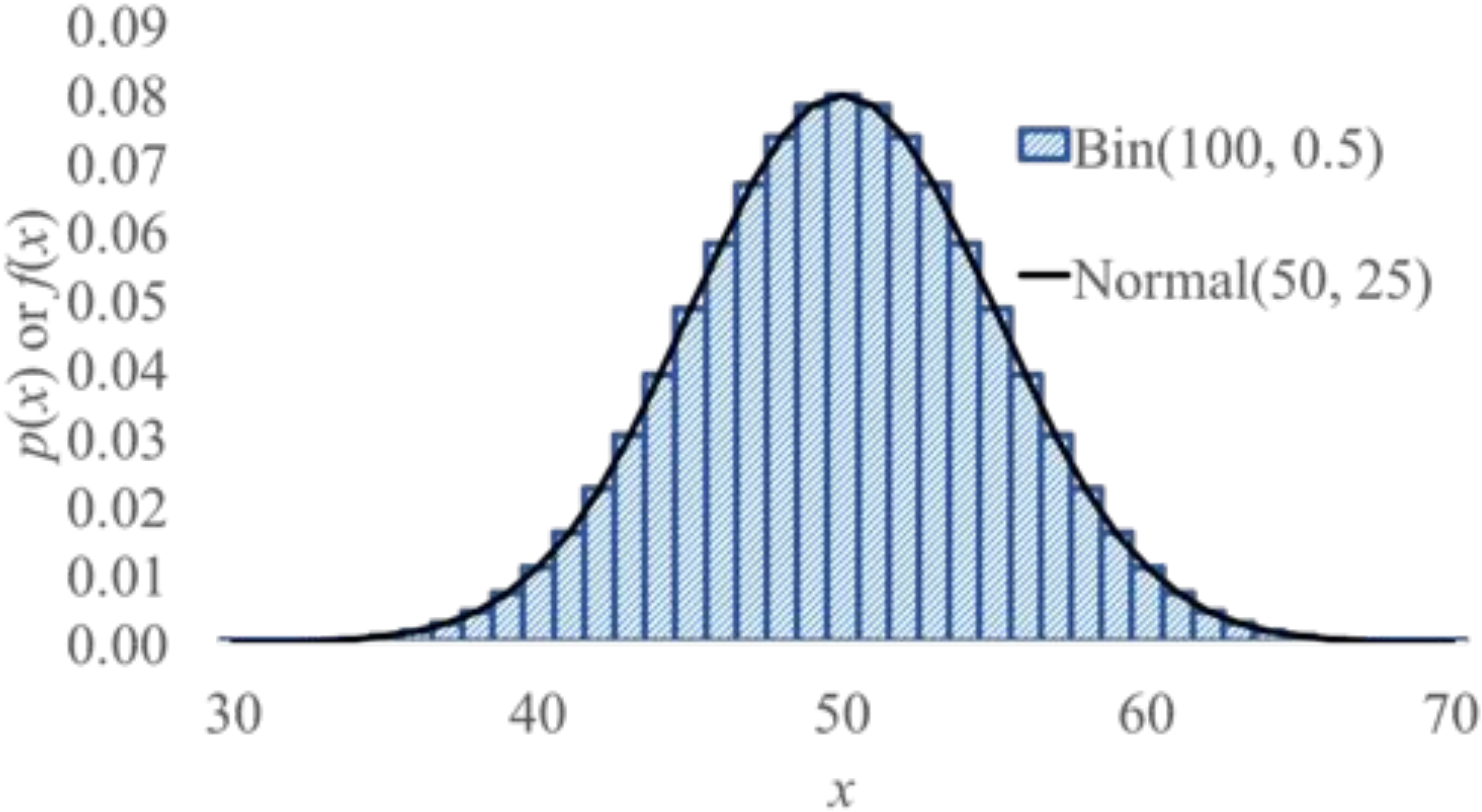
Website Testing

- 100 people are given a new website design
  - X = # people whose time on site increases
  - CEO will endorse new design if X ≥ 65 What is P(CEO endorses change| it has no effect)?
  - X ~ Bin(100, 0.5). Want to calculate P(X ≥ 65)
  - Give a numerical answer…

$$P(X \geq 65) = \sum_{i=65}^{100} \binom{100}{i} (0.5)^i (1-0.5)^{100-i}$$

Normal Approximates Binomial

Normal Approximates Binomial



Let’s invent an approximation!

Website Testing

- 100 people are given a new website design
  - X = # people whose time on site increases
  - CEO will endorse new design if X ≥ 65 What is P(CEO endorses change| it has no effect)?
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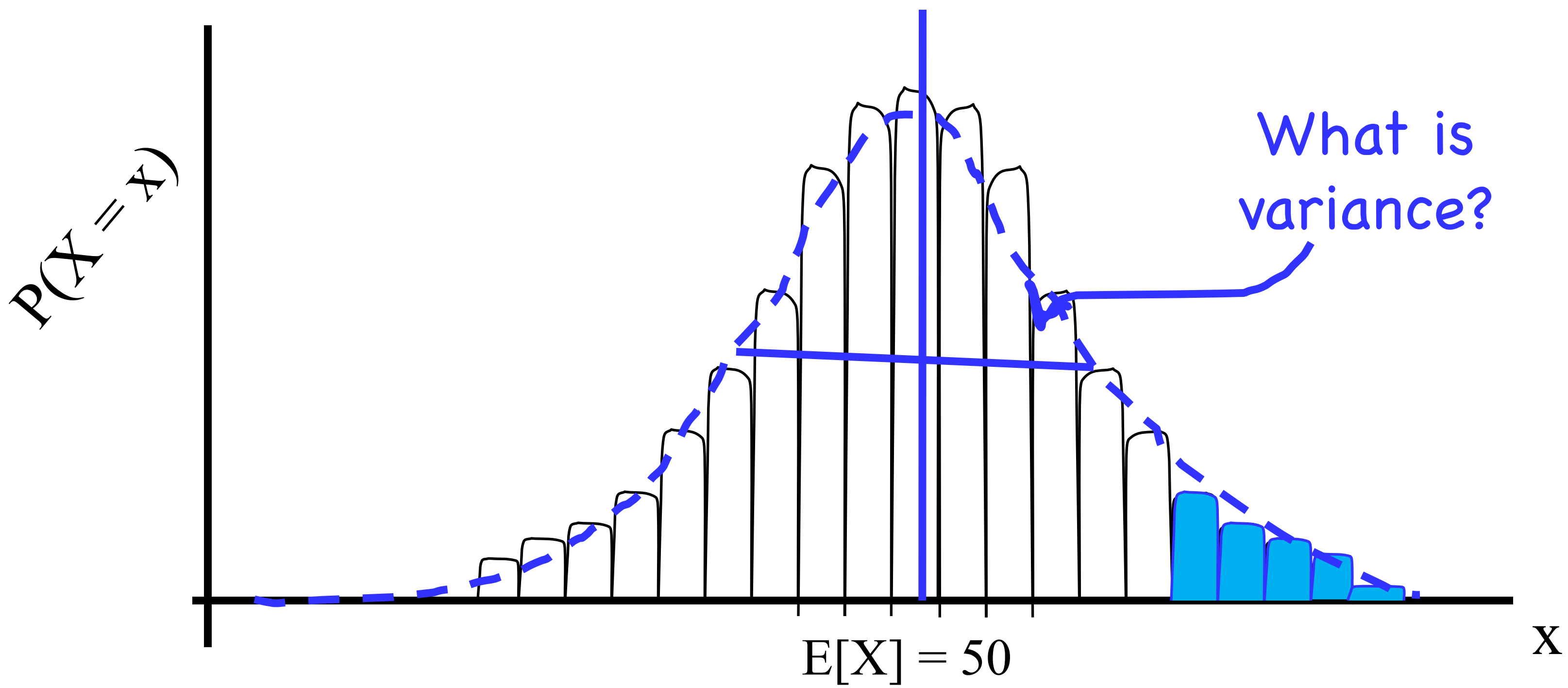


Website Testing

- 100 people are given a new website design
  - X = # people whose time on site increases
  - CEO will endorse new design if X ≥ 65 What is P(CEO endorses change| it has no effect)?
  - X ~ Bin(100, 0.5). Want to calculate P(X ≥ 65)

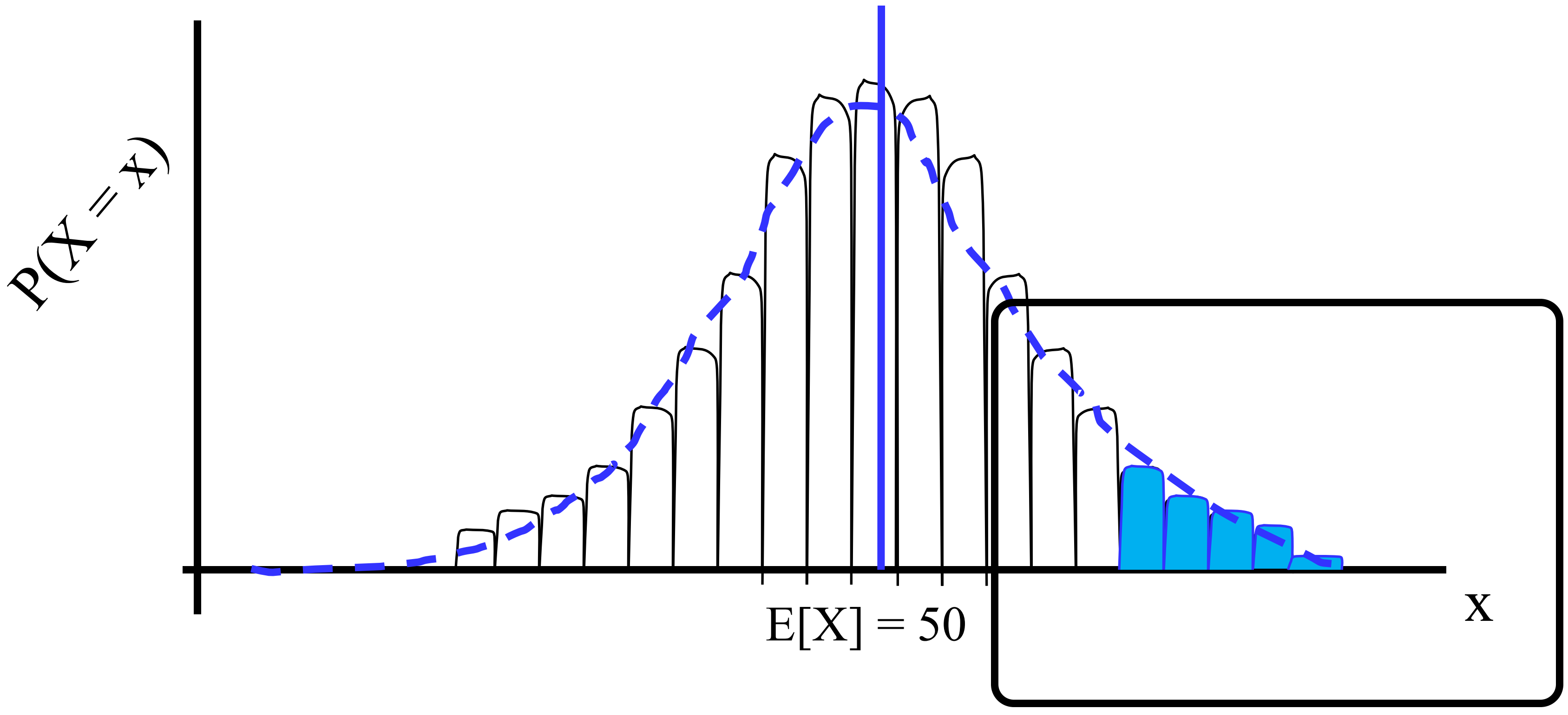
$$np = 50 \quad np(1-p) = 25 \quad \sqrt{np(1-p)} = 5$$

  - Use Normal approximation: Y ~ N(50, 25)

$$P(Y \geq 65) = P(\frac{Y-50}{5} > \frac{65-50}{5}) = P(Z > 3) = 1 - \phi(3) \approx 0.0013$$

  - Using Binomial: $P(X \geq 65) \approx 0.0018$

Website Testing



Continuity Correction

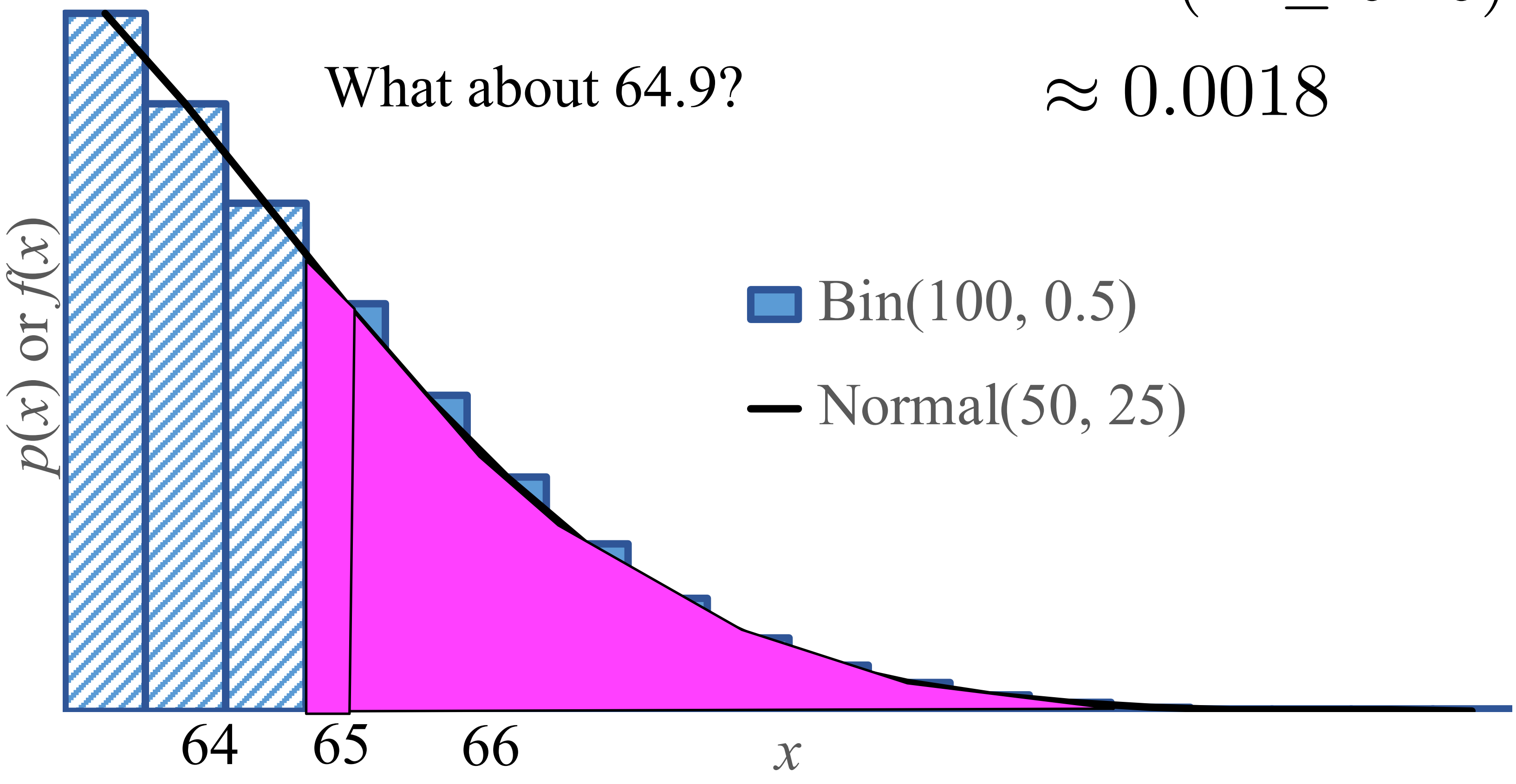
If Y (normal) approximates X (binomial)

$$P(X \geq 65)$$

$$\approx P(Y \geq 64.5)$$

$$\approx 0.0018$$

What about 64.9?



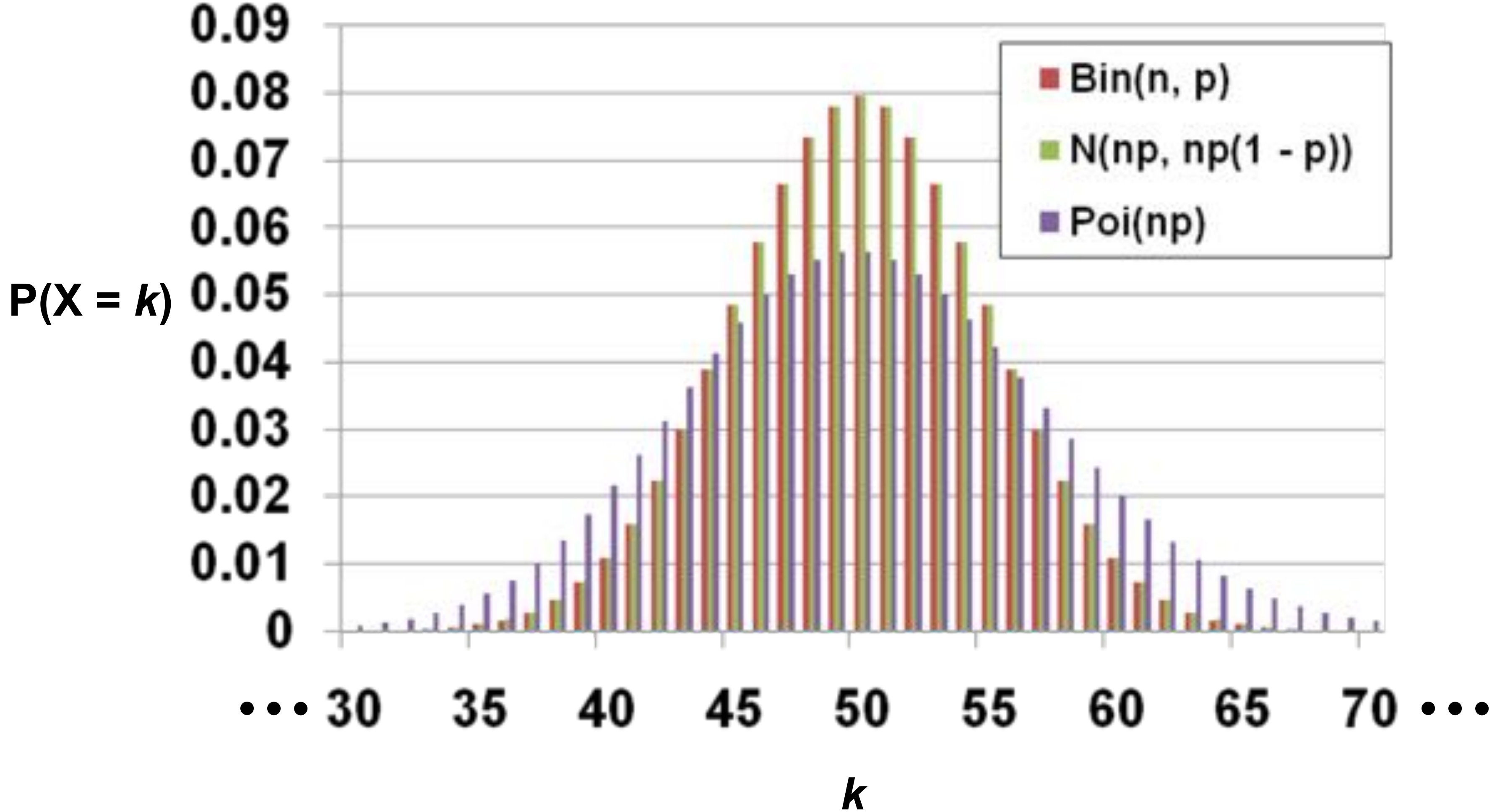
Continuity Correction

If Y (normal) approximates X (binomial)

Discrete (eg Binomial) probability question	Continuous (Normal) probability question
$X = 6$	$5.5 < Y < 6.5$
$X >= 6$	$Y > 5.5$
$X > 6$	$Y > 6.5$
$X < 6$	$Y < 5.5$
$X <= 6$	$Y < 6.5$

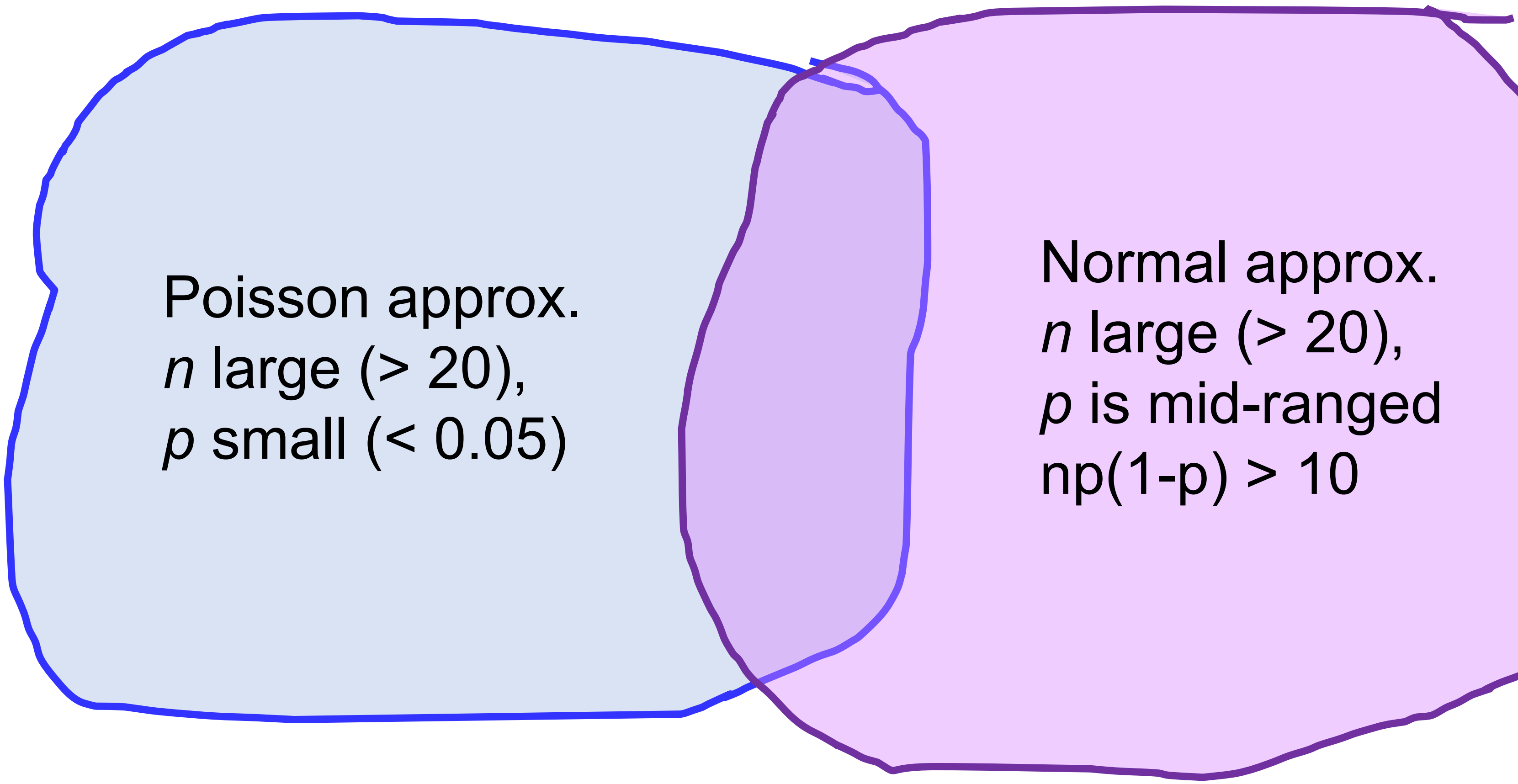
* Note: Binomial is always defined in units of “1”

Comparison when n = 100, p = 0.5



Who Gets to Approximate?

X ~ Bin(n, p)



If there is a choice, go with the normal approximation